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Cornelis van der Mee**

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# EXACT SOLUTIONS TO THE FOCUSING NONLINEAR SCHRÖDINGER EQUATION

Tuncay Aktosun

Department of Mathematics  
University of Texas at Arlington  
Arlington, TX 76019-0408, USA

Francesco Demontis and Cornelis van der Mee

Dipartimento di Matematica e Informatica  
Università di Cagliari  
Viale Merello 92, 09123 Cagliari, Italy

**Abstract:** A method is given to construct globally analytic (in space and time) exact solutions to the focusing cubic nonlinear Schrödinger equation on the line. An explicit formula and its equivalents are presented to express such exact solutions in a compact form in terms of matrix exponentials. Such exact solutions can alternatively be written explicitly as algebraic combinations of exponential, trigonometric, and polynomial functions of the spatial and temporal coordinates.

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**Short title:** Exact solutions to the NLS equation

## 1. INTRODUCTION

Consider the focusing cubic nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2u = 0, \tag{1.1}$$

where the subscripts denote the appropriate partial derivatives. The NLS equation is important for many reasons [1-3,5,31,37]. It arises in many application areas such as wave propagation in nonlinear media [37], surface waves on sufficiently deep waters [36], and signal propagation in optical fibers [24-26]. It was also the second nonlinear partial differential equation (PDE) whose initial value problem was discovered [37] to be solvable via the inverse scattering transform (IST) method.

In this paper we present a method to construct certain exact solutions to (1.1) that are globally analytic on the entire  $xt$ -plane and that decay exponentially as  $x \rightarrow \pm\infty$  at each fixed  $t \in \mathbf{R}$ . We derive an explicit formula, namely (4.11), and its equivalents (4.12), (5.14), and (6.9), in order to write such solutions in a compact form utilizing matrix exponentials. These solutions can alternatively be expressed explicitly as algebraic combinations of exponential, trigonometric, and polynomial functions of  $x$  and  $t$ . We also present an explicit formula, namely (5.6), and its equivalents (6.14) and (6.15), for the magnitude of such solutions.

The idea behind our method is similar to that used in [10] to generate exact solutions to the Korteweg-de Vries equation on the half line, and we are motivated by the use of the IST with rational scattering data. This involves representing the corresponding scattering data in terms of a matrix realization [11], establishing the separability of the kernel of a related Marchenko integral equation by expressing that kernel in terms of a matrix exponential, solving the Marchenko integral equation algebraically, and observing that the procedure leads to exact solutions to the NLS equation even when the input to the Marchenko equation does not necessarily come from any scattering data.

For the general use of rational scattering data in inverse scattering theory, the reader is referred, for example, to [8,9,17] and the references therein.

Our method has several advantages:

- (i) It is generalizable to obtain similar explicit formulas for exact solutions to other integrable nonlinear PDEs where the IST involves the use of a Marchenko integral equation. For example, a similar method has been used [10] for the half-line Korteweg-de Vries equation, and it can be applied to other equations such as the defocusing nonlinear Schrödinger equation, the modified Korteweg-de Vries equation, and the sine-Gordon equation.
- (ii) It is generalizable to the matrix versions of the aforementioned integrable nonlinear PDEs. For example, a similar method has been applied in the second author's Ph.D. thesis [20] to the matrix NLS equation in the focusing case with a cubic nonlinearity.
- (iii) As seen from our explicit formula (4.11), our exact solutions are represented in a simple and compact form in terms of a square matrix  $A$ , a constant row vector  $C$ , and a constant column vector  $B$ , where  $A$  appears in a matrix exponential. Such matrix exponentials can be “unpacked” in a straightforward way to express our exact solutions in terms of exponential, trigonometric, and polynomial functions. Depending on the size of  $A$ , such unpacked expressions may take many pages to display. Our explicit formula and its equivalents allow easy evaluation of such unpacked expressions and numerical evaluations on such exact solutions, as evident from the examples in available Mathematica notebooks [39].
- (iv) Our method easily deals with nonsimple bound-state poles and the time evolution of the corresponding bound-state norming constants. In the literature, nonsimple bound-state poles are usually avoided due to mathematical complications. We refer the reader to [32], where nonsimple bound-state poles are investigated and complications are encountered. A systematic treatment of nonsimple bound states has recently been

given [13].

- (v) Our method might be generalizable to the case where the matrix  $A$  becomes a linear operator on a separable Hilbert space. Such a generalization on which we are currently working would allow us to solve the NLS equation with initial potentials more general than those considered in our paper.

Our method to produce exact solutions to the NLS equation is based on using the IST [1-3,5,31,37]. There are also other methods to obtain solutions to (1.1). Such methods include the use of a Darboux transformation [16], the use of a Bäcklund transformation [12,14], the bilinear method of Hirota [28], the use of various other transformations such as the Hasimoto transformation [15,27], and various other techniques [6] based on guessing the form of a solution and adjusting various parameters. The main idea behind using the transformations of Darboux and Bäcklund is to produce new solutions to (1.1) from previously known solutions, and other transformations are used to produce solutions to the NLS equation from solutions to other integrable PDEs. The basic idea behind the method of Hirota is to represent the solution as a ratio of two functions and to determine those two functions by solving some corresponding coupled differential equations. A unified treatment of Hirota's method, the IST, and Bäcklund transformation to obtain soliton solutions with simple and multiple poles for the Sine-Gordon equation was given by Pöppe by using Fredholm determinants [33]. Other techniques may use an ansatz such as determining  $\Theta(x, t)$  and  $M(x, t)$  by using  $u(x, t) = e^{i\Theta(x, t)} M(x, t)$  in (1.1). For example, trying

$$u(x, t) = e^{i(k_1 x + k_2 t + k_3)} f(k_4 x + k_5 t + k_6), \quad (1.2)$$

where  $k_j$  are constant real parameters and  $f$  is a real-valued smooth function, we get an exact solution if we choose  $k_2 = 1 - k_1^2$ ,  $k_4 = \pm 1$ ,  $k_5 = \mp 2k_1$ , and  $f$  as the hyperbolic secant. One can also use the fact that if  $U(x, t)$  is a solution to (1.1), so is  $e^{ic(x-ct)} U(x - 2ct, t)$  for any real constant  $c$ . Multiplying a solution by a complex constant of unit amplitude yields another solution, and hence such a phase factor can always be omitted from the solution.

There are many references in which some exact solutions to (1.1) are presented. For example, [38] lists five explicit solutions, one is of the form of (1.2) with a constant  $f$ , the second and third with  $f$  as the hyperbolic secant (those are one-soliton solutions with simple poles), the fourth being periodic in  $x$ , and the fifth is the  $n$ -soliton solution. Another solution, which is periodic in  $x$ , is [6]

$$u(x, t) = ae^{2ia^2t} \left[ \frac{2b^2 \cosh(2a^2b\sqrt{2-b^2}t) + 2ib\sqrt{2-b^2} \sinh(2a^2b\sqrt{2-b^2}t)}{2 \cosh(2a^2b\sqrt{2-b^2}t) - \sqrt{2}\sqrt{2-b^2} \cos(\sqrt{2}abx)} - 1 \right], \quad (1.3)$$

where  $a$  and  $b$  are arbitrary real parameters. By letting  $b \rightarrow 0$  in (1.3) we get the solution

$$u(x, t) = ae^{2ia^2t} \frac{3 + 16ia^2t - 16a^4t^2 - 4a^2x^2}{1 + 16a^4t^2 + 4a^2x^2}.$$

Another exact solution which is periodic in  $x$  is presented [7] in terms of the Jacobi elliptic functions. An exact solution to (1.1) is displayed [22] in the form of a specific matrix realization and is shown to be valid for  $t \in [0, \epsilon)$  for some small  $\epsilon$  and  $x \in [0, +\infty)$ . In their celebrated paper [37] Zakharov and Shabat list the one- and  $n$ -soliton solutions as well as a one-soliton solution with a double pole, which is obtained from a two-soliton solution with simple poles by letting those poles coalesce. In [32] solitons with multiple eigenvalues are analyzed and a one-soliton solution with a double pole and a one-soliton solution with a triple pole are listed with the help of the symbolic software REDUCE, by stating that “in an actual calculation it is very complex to exceed” higher order poles. With our method in this paper we show that such solitons with any number of poles and any multiplicities can be easily expressed by using an appropriate representation. Let us also add that some periodic or almost periodic solutions can be obtained in terms of two hyperelliptic theta functions [29,30], and the scattering data for (2.1) can be constructed corresponding to certain initial profiles [34,35].

In order to appreciate the power of our method, to see why it produces new solutions, and to understand why it produces exact solutions that are either impossible or difficult to produce by other methods, let us consider the following. When the matrix size is

large (imagine  $A$  being a  $1000 \times 1000$  matrix) we have an explicit compact formula for an exact solution as in (4.11) or its equivalents (4.12), (5.14), and (6.9). By using a computer algebra system we can explicitly express such a solution in terms of exponential, trigonometric, and polynomial functions of  $x$  and  $t$  (even though such an expression will take thousands of pages to display, we are able to write such an expression thanks to our explicit formula). The only explicit formula in the literature comparable to ours is the formula for the  $n$ -soliton solution without multiplicities. Our own explicit formula yields that explicit  $n$ -soliton solution without multiplicities in a trivial case; namely, when  $A$  is a diagonal matrix of distinct entries with positive real parts, as indicated in (7.1). Our explicit formula also easily yields the  $n$ -soliton solution with arbitrary multiplicities as a special case. Dealing with even a single soliton with multiplicities has not been an easy task in other methods; for example, the exact solution example presented in [37] for a one-soliton solution with a double pole, which is obtained by coalescing two distinct poles into one, contains a typographical error, as pointed out in [32].

Our paper is organized as follows. In Section 2 we present the preliminaries and outline the Marchenko method to solve the inverse scattering problem for the Zakharov-Shabat system given in (2.1), summarize the IST for the NLS equation, and list in (2.12) the time evolution of the norming constants in a compact form [13], which is valid even when bound-state poles may have multiplicities greater than one. In Section 3 we consider (2.1) with some rational scattering data, which in turn we express in terms of the matrices  $A$ ,  $B$ ,  $C$  given in (3.5)-(3.7), respectively. In Section 4, we derive the explicit formula (4.11) for our exact solutions  $u(x, t)$  to (1.1) in terms of  $A$ ,  $B$ ,  $C$ , and we show that such solutions have analytic extensions to the entire  $xt$ -plane when the real parts of the eigenvalues of  $A$  are positive. In Section 5 we independently and directly verify that (4.11) is a solution to (1.1) as long as the matrix  $\Gamma(x; t)$  given in (4.7) is invertible, which is assured on the entire  $xt$ -plane when the real parts of the eigenvalues of  $A$  are positive. In Section 5 we also show that  $|u(x, t)|^2$  can be expressed in terms of the logarithmic derivative of the

determinant of  $\Gamma(x; t)$ . In Section 6 we remove the positivity restriction on the real parts of the eigenvalues of  $A$ , and we enlarge the class of exact solutions represented by our explicit formula (4.11) or its equivalents (4.12), (5.14), and (6.9). Finally, in Section 7 we present some examples showing how our explicit formula easily yields exact solutions to (1.1) expressed in terms of exponential, trigonometric, and polynomial functions, and we also mention the availability of various Mathematica notebooks [39], in which the user can easily modify the input and produce various exact solutions to (1.1) and their animations by specifying  $A$ ,  $B$ ,  $C$ .

## 2. PRELIMINARIES

Consider the Zakharov-Shabat system on the full line

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix}' = \begin{bmatrix} -i\lambda & q(x) \\ -\overline{q(x)} & i\lambda \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad x \in \mathbf{R}, \quad (2.1)$$

where the prime denotes the  $x$ -derivative,  $\lambda$  is the complex-valued spectral parameter,  $q$  is a complex-valued integrable potential, and the bar denotes complex conjugation. There are two linearly independent vector solutions to (2.1) denoted by  $\psi(\lambda, x)$  and  $\phi(\lambda, x)$ , which are usually known as the Jost solutions and are uniquely obtained by imposing the respective asymptotic conditions

$$\psi(\lambda, x) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \quad x \rightarrow +\infty, \quad (2.2)$$

$$\phi(\lambda, x) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \quad x \rightarrow -\infty.$$

The transmission coefficient  $T$ , the left reflection coefficient  $L$ , and the right reflection coefficient  $R$  are then obtained through the asymptotics

$$\psi(\lambda, x) = \begin{bmatrix} e^{-i\lambda x} L(\lambda)/T(\lambda) \\ e^{i\lambda x}/T(\lambda) \end{bmatrix} + o(1), \quad x \rightarrow -\infty, \quad (2.3)$$

$$\phi(\lambda, x) = \begin{bmatrix} e^{-i\lambda x}/T(\lambda) \\ e^{i\lambda x} R(\lambda)/T(\lambda) \end{bmatrix} + o(1), \quad x \rightarrow +\infty. \quad (2.4)$$



For further information on these scattering solutions to (2.1) we refer the reader to [1-3,5,31,37] and the references therein.

Besides scattering solutions to (2.1), we have so-called bound-state solutions, which are square-integrable solutions to (2.1). They occur at the poles of  $T$  in the upper half complex plane  $\mathbf{C}^+$ . Let us denote the (distinct) bound-state poles of  $T$  by  $\lambda_j$  for  $j = m+1, \dots, m+n$ , and suppose that the multiplicity of the pole at  $\lambda_j$  is given by  $n_j$ . The reason to start indexing the bound states with  $j = m+1$  instead of  $j = 1$  is for notational convenience. It is known [1-3,5,31,37] that there is only one linearly independent square-integrable vector solution to (2.1) when  $\lambda = \lambda_j$  for  $j = m+1, \dots, m+n$ . Associated with each such  $\lambda_j$ , we have  $n_j$  bound-state norming constants  $c_{js}$  for  $s = 0, \dots, n_j - 1$ .

The inverse scattering problem for (2.1) consists of recovery of  $q(x)$  for  $x \in \mathbf{R}$  from an appropriate set of scattering data such as the one consisting of the reflection coefficient  $R(\lambda)$  for  $\lambda \in \mathbf{R}$  and the bound-state information  $\{\lambda_j, \{c_{js}\}_{s=0}^{n_j-1}\}_{j=m+1}^{m+n}$ . This problem can be solved via the Marchenko method as follows [1-3,5,31,37]:

- a) From the scattering data  $\{R(\lambda), \{\lambda_j\}, \{c_{js}\}\}$ , form the Marchenko kernel  $\Omega$  as

$$\Omega(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda R(\lambda) e^{i\lambda y} + \sum_{j=m+1}^{m+n} \sum_{s=0}^{n_j-1} c_{js} \frac{y^s}{s!} e^{i\lambda_j y}. \quad (2.5)$$

- b) Solve the Marchenko equation

$$K(x, y) - \overline{\Omega(x+y)} + \int_x^{\infty} dz \int_x^{\infty} ds K(x, s) \Omega(s+z) \overline{\Omega(z+y)} = 0, \quad y > x. \quad (2.6)$$

- c) Recover the potential  $q$  from the solution  $K(x, y)$  to the Marchenko equation via

$$q(x) = -2K(x, x). \quad (2.7)$$

- d) Having determined  $K(x, y)$ , also determine

$$G(x, y) := - \int_x^{\infty} dz \overline{K(x, z)} \overline{\Omega(z+y)}. \quad (2.8)$$

Then, obtain the Jost solution  $\psi(\lambda, x)$  to the Zakharov-Shabat system (2.1)-(2.2) via

$$\psi(\lambda, x) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + \int_x^\infty dy \begin{bmatrix} K(x, y) \\ G(x, y) \end{bmatrix} e^{i\lambda y}. \quad (2.9)$$

Note that  $|q(x)|^2$  can be calculated from (2.7) or equivalently by using [37]

$$\int_x^\infty dz |q(z)|^2 = -2G(x, x), \quad |q(x)|^2 = 2 \frac{dG(x, x)}{dx}. \quad (2.10)$$

The initial-value problem for (1.1) consists of recovery of  $u(x, t)$  for  $t > 0$  when  $u(x, 0)$  is available. When  $u(x, 0) = q(x)$ , where  $q$  is the potential appearing in (2.1), it is known that such an initial-value problem can be solved [1-3,5,31,37] by the method of IST as indicated in the following diagram:

$$\begin{array}{ccc} u(x, 0) & \xrightarrow{\text{direct scattering}} & \{R(\lambda), \{\lambda_j\}, \{c_{js}\}\} \\ \text{solution to NLS} \downarrow & & \downarrow \text{time evolution} \\ u(x, t) & \xleftarrow{\text{inverse scattering}} & \{R(\lambda; t), \{\lambda_j\}, \{c_{js}(t)\}\} \end{array}$$

The application of the IST involves three steps:

- i) Corresponding to the initial potential  $q(x)$ , obtain the scattering data at  $t = 0$ ; namely, the reflection coefficient  $R(\lambda)$ , the bound-state poles  $\lambda_j$  of  $T(\lambda)$ , and the norming constants  $c_{js}$ .
- ii) Let the initial scattering data evolve in time. The time-evolved reflection coefficient  $R(\lambda; t)$  is obtained from the reflection coefficient  $R(\lambda)$  via

$$R(\lambda; t) = R(\lambda) e^{4i\lambda^2 t}. \quad (2.11)$$

The bound-state poles  $\lambda_j$  and  $T(\lambda)$  do not change in time. The time evolution of the bound-state norming constants  $c_{js}(t)$  has been known when  $s = 0$  as

$$c_{j0}(t) = c_{j0} e^{4i\lambda_j^2 t}, \quad j = n + 1, \dots, m + n.$$

The time evolution of the remaining terms has recently been analyzed in a systematic way [13], and the evolution of  $c_{js}(t)$  is described by the product of  $e^{4i\lambda_j^2 t}$  and a polynomial in  $t$  of order  $s$ ; we have [13]

$$[c_{j(n_j-1)}(t) \quad \dots \quad c_{j0}(t)] = [c_{j(n_j-1)} \quad \dots \quad c_{j0}] e^{-4iA_j^2 t}, \quad (2.12)$$

where  $A_j$  is the matrix defined in (3.3). See also [32], where a more complicated procedure is given to obtain  $c_{js}(t)$ .

- iii) Solve the inverse scattering problem for (2.1) with the time-evolved scattering data  $\{R(\lambda; t), \{\lambda_j, \{c_{js}(t)\}_{s=0}^{n_j-1}\}_{j=m+1}^{m+n}\}$  in order to obtain the time-evolved potential. It turns out that the resulting time-evolved potential  $u(x, t)$  is a solution to (1.1) and reduces to  $q(x)$  at  $t = 0$ . This inverse problem can be solved by the Marchenko method as outlined in Section 4 by replacing the kernel  $\Omega(y)$  with its time-evolved version  $\Omega(y; t)$ , which is obtained by replacing in (2.5)  $R(\lambda)$  by  $R(\lambda; t)$  and  $c_{js}$  by  $c_{js}(t)$ .

### 3. REPRESENTATION OF THE SCATTERING DATA

We are interested in obtaining explicit solutions to (1.1) when the reflection coefficient  $R(\lambda)$  appearing in (2.4) is a rational function of  $\lambda$  with poles occurring in  $\mathbf{C}^+$ . For this purpose we will use a method similar to the one developed in [10] and already applied to the half-line Korteweg-de Vries equation. We will first represent our scattering data in terms of a constant square matrix  $A$ , a constant column vector  $B$ , and a constant row vector  $C$ . We will then rewrite the Marchenko kernel  $\Omega(y)$  given in (2.5) in terms of  $A, B, C$ . It will turn out that the time-evolved kernel  $\Omega(y; t)$  will be related to  $\Omega(y)$  in an easy manner. By solving the Marchenko equation (2.6) with the time-evolved kernel  $\Omega(y; t)$ , we will obtain the time-evolved solution  $K(x, y; t)$ , from which we will recover the time-evolved potential  $u(x, t)$  in a manner analogous to (2.7).

In this section we show how to construct  $A, B, C$  from some rational scattering data

associated with the Zakharov-Shabat system. We show that our exact solutions can be obtained by choosing our triplet  $A, B, C$  as in (3.5)-(3.7), where  $\lambda_j$  are distinct and  $c_{j(n_j-1)} \neq 0$  for  $j = 1, \dots, m+n$ .

When the rational  $R(\lambda)$  has poles at  $\lambda_j$  in  $\mathbf{C}^+$  with multiplicity  $n_j$  for  $j = 1, \dots, m$ , since  $R(\lambda) \rightarrow 0$  as  $\lambda \rightarrow \infty$ , the partial fraction expansion of  $R(\lambda)$  can be written as

$$R(\lambda) = \sum_{j=1}^m \sum_{s=1}^{n_j} \frac{(-i)^s r_{js}}{(\lambda - \lambda_j)^s}, \quad (3.1)$$

for some complex coefficients  $r_{js}$ . Note that we can represent the inner summation in (3.1) in the form

$$\sum_{s=1}^{n_j} \frac{(-i)^s r_{js}}{(\lambda - \lambda_j)^s} = -i C_j (\lambda - i A_j)^{-1} B_j, \quad (3.2)$$

where, for  $j = 1, \dots, m$ , we have defined

$$A_j := \begin{bmatrix} -i\lambda_j & -1 & 0 & \dots & 0 & 0 \\ 0 & -i\lambda_j & -1 & \dots & 0 & 0 \\ 0 & 0 & -i\lambda_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -i\lambda_j & -1 \\ 0 & 0 & 0 & \dots & 0 & -i\lambda_j \end{bmatrix}, \quad B_j := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad (3.3)$$

$$C_j := [r_{jn_j} \quad \dots \quad r_{j1}],$$

so that

$$\lambda - i A_j = \begin{bmatrix} \lambda - \lambda_j & i & 0 & \dots & 0 & 0 \\ 0 & \lambda - \lambda_j & i & \dots & 0 & 0 \\ 0 & 0 & \lambda - \lambda_j & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda - \lambda_j & i \\ 0 & 0 & 0 & \dots & 0 & \lambda - \lambda_j \end{bmatrix},$$

$$(\lambda - i A_j)^{-1} = \begin{bmatrix} \frac{1}{\lambda - \lambda_j} & \frac{-i}{(\lambda - \lambda_j)^2} & \frac{(-i)^2}{(\lambda - \lambda_j)^3} & \dots & \frac{(-i)^{n_j-2}}{(\lambda - \lambda_j)^{n_j-1}} & \frac{(-i)^{n_j-1}}{(\lambda - \lambda_j)^{n_j}} \\ 0 & \frac{1}{\lambda - \lambda_j} & \frac{-i}{(\lambda - \lambda_j)^2} & \dots & \frac{(-i)^{n_j-3}}{(\lambda - \lambda_j)^{n_j-2}} & \frac{(-i)^{n_j-2}}{(\lambda - \lambda_j)^{n_j-1}} \\ 0 & 0 & \frac{1}{\lambda - \lambda_j} & \dots & \frac{(-i)^{n_j-4}}{(\lambda - \lambda_j)^{n_j-3}} & \frac{(-i)^{n_j-3}}{(\lambda - \lambda_j)^{n_j-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\lambda - \lambda_j} & \frac{-i}{(\lambda - \lambda_j)^2} \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{\lambda - \lambda_j} \end{bmatrix}.$$

We remark that the row vector  $C_j$  contains  $n_j$  entries, the column vector  $B_j$  contains  $n_j$  entries, and  $A_j$  is an  $n_j \times n_j$  square matrix,  $(-A_j)$  is in a Jordan canonical form, and that  $(\lambda - iA_j)^{-1}$  is an upper triangular Toeplitz matrix.

As for the bound states, for  $j = m+1, \dots, m+n$ , let us use (3.3) to define the  $n_j \times n_j$  matrix  $A_j$  and the column  $n_j$ -vector  $B_j$ , and let  $C_j$  be the row  $n_j$ -vector defined as

$$C_j := [c_{j(n_j-1)} \quad \dots \quad c_{j0}],$$

so that the summation term in (2.5) is obtained as

$$\sum_{s=0}^{n_j-1} c_{js} \frac{y^s}{s!} e^{i\lambda_j y} = -\frac{i}{2\pi} \int_{-\infty}^{\infty} d\lambda C_j (\lambda - iA_j)^{-1} B_j e^{i\lambda y}, \quad y > 0. \quad (3.4)$$

Now let us define the  $p \times p$  block diagonal matrix  $A$  as

$$A := \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{m+n} \end{bmatrix}, \quad (3.5)$$

where  $p$  is the integer given by

$$p := \sum_{j=1}^{m+n} n_j.$$

Similarly, let us define the column  $p$ -vector  $B$  as

$$B := \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{m+n} \end{bmatrix}, \quad (3.6)$$

and the row  $p$ -vector  $C$  as

$$C := [C_1 \quad C_2 \quad \dots \quad C_{m+n}]. \quad (3.7)$$

Without loss of generality we can assume that  $\lambda_j$  for  $j = 1, \dots, m+n$  are all distinct; in case one of  $\lambda_j$  for  $j = 1, \dots, m$  coincides with one of  $\lambda_j$  for  $j = m+1, \dots, m+n$ , we

can simply combine the corresponding blocks in (3.5) to reduce the number of blocks in  $A$  by one. In case more such  $\lambda_j$  coincide, we can proceed in a similar way so that each block in (3.5) will be associated with a distinct  $\lambda_j$ . Similarly, we can combine the corresponding blocks in each of (3.6) and (3.9) so that the sizes of  $B$  and  $C$  will be compatible with the size of  $A$ .

Consider the function  $P(\lambda)$  defined as

$$P(\lambda) := -iC(\lambda - iA)^{-1}B, \quad \lambda \in \mathbf{C}, \quad (3.8)$$

with the triplet  $A, B, C$ , where the constant matrices  $A, B, C$  have sizes  $p \times p, p \times 1$ , and  $1 \times p$ , respectively, and the singularities of  $P(\lambda)$  occur at the eigenvalues of  $iA$ . Such a representation is called *minimal* [11] if there do not exist constant matrices  $\tilde{A}, \tilde{B}, \tilde{C}$  with sizes  $\tilde{p} \times \tilde{p}, \tilde{p} \times 1$ , and  $1 \times \tilde{p}$ , respectively, such that  $P(\lambda) = -i\tilde{C}(\lambda - i\tilde{A})^{-1}\tilde{B}$  and  $\tilde{p} < p$ . There always exists a triplet corresponding to a minimal representation. It is known [11] that the realization with the triplet  $A, B, C$  is minimal if and only if the two  $p \times p$  matrices defined as

$$\text{col}_p(C, A) := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix}, \quad \text{row}_p(A, B) := [B \quad AB \quad \dots \quad A^{p-1}B], \quad (3.9)$$

both have rank  $p$ .

The following theorem shows that, for the sake of constructing exact solutions to (1.1), it is sufficient to consider only the triplet  $A, B, C$  given in (3.5)-(3.7) with distinct  $\lambda_j$  for  $j = 1, \dots, m + n$  because any other triplet  $\tilde{A}, \tilde{B}, \tilde{C}$  with sizes  $p \times p, p \times 1$ , and  $1 \times p$ , respectively, can be equivalently expressed in terms of  $A, B, C$ .

**Theorem 3.1** *Given any arbitrary triplet  $\tilde{A}, \tilde{B}, \tilde{C}$  with sizes  $p \times p, p \times 1$ , and  $1 \times p$ , respectively, there exists a triplet  $A, B, C$  having the form given in (3.5)-(3.7), respectively, which yields the same exact solution to (1.1). The construction of  $A, B, C$  can be achieved*

by using

$$\tilde{A} = MAM^{-1}, \quad \tilde{B} = MSB, \quad C = \tilde{C}MS, \quad (3.10)$$

where  $M$  is an invertible matrix whose columns consist of the generalized eigenvectors of  $\tilde{A}$ , the matrix  $S$  is an upper triangular Toeplitz matrix commuting with  $A$ , and the complex entries of  $C$  are chosen as in (3.10).

PROOF: Since  $(-A)$  is in the Jordan canonical form, any given  $\tilde{A}$  can be converted to  $A$  by using  $\tilde{A} = MAM^{-1}$ , where  $M$  is a matrix whose columns are formed by using the generalized eigenvectors of  $(-\tilde{A})$ . Next, consider all matrices  $S$  commuting with  $A$ . Any such matrix has the block diagonal form

$$S := \begin{bmatrix} S_1 & 0 & \dots & 0 \\ 0 & S_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & S_{m+n} \end{bmatrix}, \quad S_j := \begin{bmatrix} \alpha_{jn_j} & \alpha_{j(n_j-1)} & \dots & \alpha_{j1} \\ 0 & \alpha_{jn_j} & \dots & \alpha_{j2} \\ 0 & 0 & \dots & \alpha_{j3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_{jn_j} \end{bmatrix}, \quad (3.11)$$

where  $n_j$  is the order of the pole  $\lambda_j$  for  $j = 1, \dots, m+n$ , and the constants  $\alpha_{js}$  are arbitrary. We will determine such  $\alpha_{js}$  and hence  $S$  itself by using  $M^{-1}\tilde{B} = SB$ . Note that  $SB$  is the column  $p$ -vector consisting of  $m+n$  column blocks, where the  $j$ th block has entries  $\alpha_{j1}, \dots, \alpha_{jn_j}$ . Thus,  $S$  is unambiguously constructed from  $M$  and  $\tilde{B}$ . Having constructed  $M$  and  $S$  from  $\tilde{A}$  and  $\tilde{B}$ , we finally choose the complex entries in the matrix  $C$  appearing in (3.7) so that  $C = \tilde{C}MS$ . Let us now show the equivalence of the representation with the triplet  $\tilde{A}, \tilde{B}, \tilde{C}$  and that with the triplet  $A, B, C$ . From (3.8) we see that we must show

$$-iC(\lambda - iA)^{-1}B = -i\tilde{C}(\lambda - i\tilde{A})^{-1}\tilde{B}. \quad (3.12)$$

Since  $SA = AS$  and  $MA = \tilde{A}M$ , we also have

$$S(\lambda - iA)^{-1} = (\lambda - iA)^{-1}S, \quad M(\lambda - iA)^{-1} = (\lambda - i\tilde{A})^{-1}M. \quad (3.13)$$

Replacing  $C$  by  $\tilde{C}MS$  on the left hand side of (3.12) and using (3.13), we establish the equality in (3.12). Similarly, replacing  $C$  by  $\tilde{C}MS$  on the right hand side of (4.2) and

using  $MA = \tilde{A}M$  and  $SA = AS$  and (3.13), we prove that  $\Omega(y; t)$  remains unchanged if  $A, B, C$  are replaced with  $\tilde{A}, \tilde{B}, \tilde{C}$ , respectively, in (4.2). Hence the triplet  $A, B, C$  and the triplet  $\tilde{A}, \tilde{B}, \tilde{C}$  yield the same solution to (1.1). ■

Note that the invertibility of  $S$  is not needed in Theorem 3.1. On the other hand, from (3.11) it is seen that  $S$  is invertible if and only if  $\alpha_{jn_j} \neq 0$  for  $j = 1, \dots, m+n$ . In the rest of this section we will give a characterization for the minimality of the representation in (3.8) with the triplet  $A, B, C$  given in (3.5)-(3.7). We will show that as long as  $\lambda_j$  are distinct and  $c_{j(n_j-1)} \neq 0$  in (3.7) for  $j = 1, \dots, m+n$ , the triplet  $A, B, C$  given in (3.5)-(3.7) can be used to recover in the form of (4.11) our exact solutions to (1.1). First, we need a result needed in the proof of Theorem 3.3.

**Proposition 3.2** *The matrix  $\text{row}_p(A, B)$  defined in (3.9) is invertible if and only if  $\lambda_j$  for  $j = 1, \dots, m+n$  appearing in (3.5) are distinct.*

PROOF: It is enough to prove that the rows of  $\text{row}_p(A, B)$  are linearly independent if and only if  $\lambda_j$  for  $j = 1, \dots, m+n$  are distinct. We will give the proof by showing that a row-echelon equivalent matrix  $T$  defined below has linearly independent rows. Using (3.5) and (3.6) we get

$$\text{row}_p(A, B) = \begin{bmatrix} \text{row}_p(A_1, B_1) \\ \text{row}_p(A_2, B_2) \\ \vdots \\ \text{row}_p(A_{m+n}, B_{m+n}) \end{bmatrix}.$$

With the help of (3.3) we see that the  $n_j \times p$  matrix  $\text{row}_p(A_j, B_j)$  is given by

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & (-1)^{p-1} \\ 0 & 0 & 0 & \dots & (-1)^{p-2} & (-1)^{p-1}(p-1)(i\lambda_j) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & 2i\lambda_j & \dots & (-1)^{p-2}(p-2)(i\lambda_j)^{p-3} & (-1)^{p-1}(p-1)(i\lambda_j)^{p-2} \\ 1 & -i\lambda_j & (i\lambda_j)^2 & \dots & (-1)^{p-2}(i\lambda_j)^{p-2} & (-1)^{p-1}(i\lambda_j)^{p-1} \end{bmatrix},$$

where we observe the binomial expansion of  $(-i\lambda_j - 1)^s$  in the  $(s-1)$ st column. Put  $\sigma(k) := \#\{j : n_j \geq k\}$ , i.e. the number of Jordan blocks of  $A$  of size at least  $k$ . Then,



$m + n = \sigma(1) \geq \sigma(2) \geq \sigma(3) \geq \dots$ . By reordering the rows of  $\text{row}_p(A, B)$  we obtain a row-equivalent  $p \times p$  echelon matrix  $T$  such that  $T_{r1} = 0$  for  $r > \sigma(1)$ ,  $T_{r2} = 0$  for  $r > \sigma(1) + \sigma(2)$ ,  $T_{r3} = 0$  for  $r > \sigma(1) + \sigma(2) + \sigma(3)$ , etc., while the submatrices consisting of the elements  $T_{rs}$  for  $r = \sigma(1) + \dots + \sigma(k-1) + 1, \dots, \sigma(1) + \dots + \sigma(k)$  and  $s = k, k+1, \dots, p$  have the form

$$\begin{bmatrix} 1 & a_{k1}\mu_1 & a_{k2}\mu_1^2 & \dots & a_{k(k-1)}\mu_1^{p-k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{k1}\mu_{\sigma(k)} & a_{k2}\mu_{\sigma(k)}^2 & \dots & a_{k(k-1)}\mu_{\sigma(k)}^{p-k-1} \end{bmatrix}, \quad (3.14)$$

where apart from a sign, the coefficients  $a_{ks}$  are the binomial coefficients and hence nonzero, and the constants  $\mu_1, \dots, \mu_{\sigma(k)}$  correspond to a rearrangement of those of  $-i\lambda_j$  for which  $n_j \geq k$ . Since the matrix given in (3.14) can be written as the product of a Vandermonde matrix and a nonsingular diagonal matrix, its rows are linearly independent if and only if  $\lambda_j$  with  $n_j \geq k$  are distinct. From the echelon structure of the matrix  $T$  it then follows that all the rows of  $T$ , and hence the rows of  $\text{row}_p(A, B)$  are linearly independent. ■

**Theorem 3.3** *The triplet  $A, B, C$  given in (3.5)-(3.7) corresponds to a minimal representation in (3.8) if and only if  $\lambda_j$  are all distinct and  $c_{j(n_j-1)} \neq 0$  for  $j = 1, \dots, m+n$ .*

PROOF: Note that the matrix  $S$  defined in (3.11) commute with  $A$ , and we have  $SA = AS$  and  $S_j A_j = A_j S_j$  for  $j = 1, \dots, m+n$ . Let us use a particular choice for  $S_j$  by letting  $\alpha_{j1} = c_{j0}, \alpha_{j2} = c_{j1}, \dots, \alpha_{jn_j} = c_{j(n_j-1)}$ . Thus,  $S$  is invertible if and only if  $c_{j(n_j-1)} \neq 0$  for  $j = 1, \dots, m+n$ . Let us define the column  $p$ -vector  $\hat{B}$  and the row  $p$ -vector  $\hat{C}$  via  $\hat{B} = SB$  and  $\hat{C}S = C$ . As in the proof of (3.12) in Theorem 3.1 we obtain

$$-iC(\lambda - iA)^{-1}B = -i\hat{C}(\lambda - iA)^{-1}\hat{B}.$$

and hence the representation in (3.8) with the triplet  $A, B, C$  is equivalent to that with  $A, \hat{B}, \hat{C}$ . From the statement containing (3.9) it then follows that our theorem is proved if we can show that  $\text{row}_p(A, \hat{B})$  and  $\text{col}_p(\hat{C}, A)$  are both invertible if and only if  $\lambda_j$  are all distinct and  $c_{j(n_j-1)} \neq 0$  for  $j = 1, \dots, m+n$ . Below we will prove that  $\text{row}_p(A, \hat{B})$  and  $\text{col}_p(\hat{C}, A)$  are invertible if and only if  $\text{row}_p(A, B)$  and  $S$  are invertible. Our theorem then

follows from Proposition 3.2 and the fact that  $S$  is invertible if and only if  $c_{j(n_j-1)} \neq 0$  for  $j = 1, \dots, m+n$ . Since  $SA = AS$  and  $SB = \hat{B}$ , from (3.9) we obtain

$$S \operatorname{row}_p(A, B) = \operatorname{row}_p(A, SB) = \operatorname{row}_p(A, \hat{B}),$$

and hence  $\operatorname{row}_p(A, \hat{B})$  is invertible if and only if  $\operatorname{row}_p(A, B)$  and  $S$  are invertible. We complete the proof by showing that  $\operatorname{col}_p(\hat{C}, A)$  is invertible if and only if  $\operatorname{row}_p(A, B)$  is invertible. Define the  $n_j \times n_j$  matrix  $J_j$  and the  $p \times p$  matrix  $J$  as

$$J_j := \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad J := \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_{m+n} \end{bmatrix}, \quad (3.15)$$

where 1 appears along the trailing diagonal of  $J_j$ . Let us use the superscript  $T$  to denote the matrix transpose. Note that

$$J_j^{-1} = J_j, \quad J_j^T = J_j, \quad J^{-1} = J, \quad J^T = J.$$

It can be verified from (3.5) that  $JAJ = A^T$ . Using (3.6), (3.7), and (3.15), since  $\hat{C} = CS^{-1}$  we get  $\hat{C} = B^T J$ . Thus, we have

$$\left( \operatorname{col}_p(\hat{C}, A) \right)^T = \operatorname{row}_p(A^T, \hat{C}^T) = \operatorname{row}_p(A^T, JB) = J \operatorname{row}_p(A, B).$$

Since  $J$  is invertible, our proof is complete. ■

#### 4. EXPLICIT SOLUTIONS TO THE NLS EQUATION

In the previous section we have constructed  $A, B, C$  given in (3.5)-(3.7), respectively, from some rational scattering data of the Zakharov-Shabat system. In this section we solve the corresponding time-evolved Marchenko equation explicitly for  $x \geq 0$  in terms of such  $A, B, C$ . Such solutions lead to explicit solutions to (1.1) via the formula given in (4.11). We then show that such solutions have analytic extensions to the entire  $xt$ -plane if

the real parts of the eigenvalues of  $A$  are positive, which is equivalent to having  $\lambda_j \in \mathbf{C}^+$  for  $j = 1, \dots, m+n$  in (3.3). We also analyze various properties of the key matrices  $Q(x;t)$ ,  $N(x)$ , and  $\Gamma(x;t)$  that appear in (4.7)-(4.9) and that are used to construct our exact solutions.

For  $y \geq 0$ , with the help (3.2), (3.4), and a contour integration along the boundary of  $\mathbf{C}^+$ , we evaluate the kernel  $\Omega(y)$  defined in (2.5) as

$$\Omega(y) = Ce^{-Ay}B, \quad y \geq 0. \quad (4.1)$$

Note that (4.1) yields a separable kernel for the Marchenko integral equation in (2.6) because from

$$\Omega(x+y) = Ce^{-Ax}e^{-Ay}B,$$

we see that  $\Omega(x+y)$  is the Euclidean product of the row  $p$ -vector  $Ce^{-Ax}$  and the column  $p$ -vector  $e^{-Ay}B$ . As a result of this separability we are able to solve the Marchenko integral equation (2.6) exactly by algebraic means.

At this point we discuss the time evolution of the scattering data in more detail. Using (2.11) we can express the time-evolved Marchenko integral kernel as

$$\Omega(y;t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda R(\lambda) e^{4i\lambda^2 t} e^{i\lambda y} + \sum_{j=m+1}^{m+n} \sum_{s=0}^{n_j-1} c_{js}(t) \frac{y^s}{s!} e^{i\lambda_j y},$$

where  $c_{js}(t)$  satisfies (2.12). This time-evolved kernel is seen to satisfy the first order PDE

$$\Omega_t(y;t) + 4i\Omega_{yy}(y;t) = 0,$$

provided the integral  $\int_{-\infty}^{\infty} d\lambda (1+\lambda^2) |R(\lambda)|$  exists. Such PDEs for Marchenko kernels have been studied in [4] for a variety of nonlinear evolution equations and in [20] for the matrix NLS equation. Here we use (4.1) as an initial condition in solving this PDE and write

$$\Omega(y;t) = Ce^{-Ay-4iA^2t}B, \quad y \geq 0. \quad (4.2)$$

In other words,  $\Omega(y; t)$  is obtained from  $\Omega(y)$  by replacing  $C$  in (4.1) by  $Ce^{-4iA^2t}$ . Let us use a dagger to denote the matrix adjoint (complex conjugate and transpose). Since  $\Omega(y; t)$  is a scalar, its complex conjugate is the same as its adjoint and we have

$$\Omega(y; t)^\dagger = B^\dagger e^{-A^\dagger y + 4i(A^\dagger)^2 t} C^\dagger. \quad (4.3)$$

Comparing with (2.6) we obtain the time-evolved Marchenko integral equation as

$$K(x, y; t) - \Omega(x+y; t)^\dagger + \int_x^\infty dz \int_x^\infty ds K(x, s; t) \Omega(s+z; t) \Omega(z+y; t)^\dagger = 0, \quad y > x. \quad (4.4)$$

Using (4.2) and (4.3) in (4.4), we see that we can look for a solution in the form

$$K(x, y; t) = H(x; t) e^{-A^\dagger y + 4i(A^\dagger)^2 t} C^\dagger, \quad (4.5)$$

where  $H(x; t)$  is to be determined. Using (4.5) in (4.4), we obtain

$$H(x; t) \Gamma(x; t) = B^\dagger e^{-A^\dagger x}, \quad (4.6)$$

where we have defined

$$\Gamma(x; t) := I + Q(x; t) N(x), \quad (4.7)$$

with  $I$  denoting the  $p \times p$  identity matrix and

$$Q(x; t) := \int_x^\infty ds e^{-A^\dagger s + 4i(A^\dagger)^2 t} C^\dagger C e^{-As - 4iA^2 t}, \quad (4.8)$$

$$N(x) := \int_x^\infty dz e^{-Az} B B^\dagger e^{-A^\dagger z}. \quad (4.9)$$

Using (4.6) in (4.5) we can write the solution to (4.4) as

$$K(x, y; t) = B^\dagger e^{-A^\dagger x} \Gamma(x; t)^{-1} e^{-A^\dagger y + 4i(A^\dagger)^2 t} C^\dagger, \quad (4.10)$$

provided  $\Gamma(x; t)$  is invertible. We will prove the invertibility of  $\Gamma(x; t)$  in Theorem 4.2. In analogy to (2.7) we get the time-evolved potential as  $u(x, t) = -2K(x, x; t)$ , and hence the solution to (1.1) is obtained as

$$u(x, t) = -2B^\dagger e^{-A^\dagger x} \Gamma(x; t)^{-1} e^{-A^\dagger x + 4i(A^\dagger)^2 t} C^\dagger. \quad (4.11)$$

It is possible [19] to write (4.11) as the ratio of two determinants as

$$u(x, t) = \frac{\det F(x; t)}{\det \Gamma(x; t)}, \quad (4.12)$$

where the  $(p + 1) \times (p + 1)$  matrix  $F(x; t)$  is given by

$$F(x; t) := \begin{bmatrix} 0 & 2B^\dagger e^{-A^\dagger x} \\ e^{-A^\dagger x + 4i(A^\dagger)^2 t} C^\dagger & \Gamma(x; t) \end{bmatrix}.$$

We end this section by listing some useful properties of the matrices  $Q(x; t)$ ,  $N(x)$ , and  $\Gamma(x; t)$ .

**Proposition 4.1** *The matrices  $Q(x; t)$  and  $N(x)$  defined in (4.8) and (4.9), respectively, satisfy*

$$Q(x; t) = e^{-A^\dagger x + 4i(A^\dagger)^2 t} Q(0; 0) e^{-Ax - 4iA^2 t}, \quad N(x) = e^{-Ax} N(0) e^{-A^\dagger x}, \quad (4.13)$$

and the integrals in (4.8) and (4.9) converge for all  $x, t \in \mathbf{R}$  as long as all the eigenvalues of  $A$  have positive real parts.

PROOF: By replacing  $s$  and  $z$  with  $s - x$  and  $z - x$  in (4.8) and (4.9), respectively, we obtain (4.13). From (4.8) and (4.9), we then get

$$Q(0; 0) = \int_0^\infty ds [C e^{-As}]^\dagger [C e^{-As}], \quad N(0) = \int_0^\infty dz [e^{-Az} B] [e^{-Az} B]^\dagger. \quad (4.14)$$

If  $\epsilon > 0$  is chosen such that the real parts of the eigenvalues of  $A$  exceed  $\epsilon$ , then in any matrix norm  $\|\cdot\|$  we have  $\|e^{-Az}\| = O(e^{-\epsilon z})$  and  $\|e^{-A^\dagger z}\| = O(e^{-\epsilon z})$  as  $z \rightarrow +\infty$ . Hence, the integrals in (4.14) converge, and as a consequence of (4.13) the integrals in (4.8) and (4.9) converge for all  $x, t \in \mathbf{R}$ . ■

The next theorem shows that the matrix  $\Gamma(x; t)$  defined in (4.7) is invertible for all  $x, t \in \mathbf{R}$  as long as the eigenvalues of  $A$  have positive real parts. In fact, in that case  $\Gamma(x; t)$  has a positive determinant for all  $x, t \in \mathbf{R}$ .

**Theorem 4.2** *Assume that the eigenvalues of  $A$  have positive real parts. Then, for every  $x, t \in \mathbf{R}$  we have the following:*

- (i) *The matrices  $Q(x; t)$  and  $N(x)$  defined in (4.8) and (4.9), respectively, are positive and selfadjoint. Consequently, there exist unique positive selfadjoint matrices  $Q(x; t)^{1/2}$  and  $N(x)^{1/2}$  such that  $Q(x; t) = Q(x; t)^{1/2}Q(x; t)^{1/2}$  and  $N(x) = N(x)^{1/2}N(x)^{1/2}$ .*
- (ii) *The matrix  $\Gamma(x; t)$  defined in (4.7) is invertible.*
- (iii) *The determinant of  $\Gamma(x; t)$  is positive.*

PROOF: In our proof let us write  $Q$  and  $N$  for  $Q(x; t)$  and  $N(x)$ , respectively. The positivity and selfadjointness of  $Q$  and  $N$  are a direct consequence of the fact that each of the integrands in (4.8) and (4.9) can be written as the product of a matrix and its adjoint; hence [23] we have proved (i). From the Sherman-Morrison-Woodbury formula [23] it follows that

$$[I + Q^{1/2}(Q^{1/2}N)]^{-1} = I - Q^{1/2}[I + (Q^{1/2}N)Q^{1/2}]^{-1}Q^{1/2}N,$$

and hence  $(I + QN)$  is invertible if and only if  $(I + Q^{1/2}NQ^{1/2})$  is invertible; on the other hand, the latter can be written as  $[I + (Q^{1/2}N^{1/2})(Q^{1/2}N^{1/2})^\dagger]$  due to the selfadjointness of  $Q^{1/2}$  and  $N^{1/2}$ , and hence it is invertible, establishing (ii). From the two matrix identities

$$\begin{aligned} \begin{bmatrix} I & 0 \\ Q^{1/2}N & I \end{bmatrix} \begin{bmatrix} I & Q^{1/2} \\ -Q^{1/2}N & I \end{bmatrix} \begin{bmatrix} I & -Q^{1/2} \\ 0 & I \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I + Q^{1/2}NQ^{1/2} \end{bmatrix}, \\ \begin{bmatrix} I & -Q^{1/2} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & Q^{1/2} \\ -Q^{1/2}N & I \end{bmatrix} \begin{bmatrix} I & 0 \\ Q^{1/2}N & I \end{bmatrix} &= \begin{bmatrix} I + QN & 0 \\ 0 & I \end{bmatrix}, \end{aligned}$$

it follows that  $I + QN$  and  $(I + Q^{1/2}NQ^{1/2})$  have the same determinant. Thus, we have (iii) as a result of the fact that the determinant of  $[I + (Q^{1/2}N^{1/2})(Q^{1/2}N^{1/2})^\dagger]$  is positive. ■

**Proposition 4.3** *Assume that the eigenvalues of  $A$  have positive real parts. Then, for all  $x, t \in \mathbf{R}$  the matrices  $Q(x; t)$ ,  $N(x)$ ,  $\Gamma(x; t)$  defined in (4.7)-(4.9) satisfy:*

$$Q_x = -A^\dagger Q - QA, \quad N_x = -AN - NA^\dagger, \quad Q_t = 4i[(A^\dagger)^2 Q - QA^2], \quad (4.15)$$

$$\Gamma^\dagger = I + NQ, \quad \Gamma^{-1}Q = Q(\Gamma^\dagger)^{-1}, \quad (\Gamma^\dagger)^{-1}N = N\Gamma^{-1}. \quad (4.16)$$

PROOF: We obtain (4.15) from (4.13), or (4.8) and (4.9), through differentiation. Using the selfadjointness of  $Q$  and  $N$  proved in Theorem 4.2, from (4.7) we obtain (4.16). ■

**Theorem 4.4** *For every  $x, t \in \mathbf{R}$ , the matrices  $Q(x; t)$  and  $N(x)$  defined in (4.8) and (4.9), respectively, are simultaneously invertible for all  $x, t \in \mathbf{R}$  if and only if the realization in (4.1) of  $\Omega(y)$  with the triplet  $A, B, C$  is minimal and the eigenvalues of  $A$  have positive real parts.*

PROOF: From (4.13) we see that it is enough to prove that  $Q(0; 0)$  and  $N(0)$  defined in (4.14) are invertible. The integrals in (4.14) are convergent as a result of the positivity of the real parts of the eigenvalues of  $A$ . If  $Q(0; 0)g = 0$  for some vector  $g \in \mathbf{C}^p$ , then from (4.14) we see that  $Ce^{-As}g = 0$  for all  $s \geq 0$ . By analytic continuation this implies that  $Ce^{-As}g = 0$  for all  $s \in \mathbf{C}$  and hence

$$CA^k g = 0, \quad k = 0, 1, \dots \quad (4.17)$$

Similarly, if  $N(0)h = 0$  for some vector  $h \in \mathbf{C}^p$ , using (4.14) we conclude that

$$B^\dagger(A^\dagger)^k h = 0, \quad k = 0, 1, \dots \quad (4.18)$$

It is known [11] that the realization in (3.8) or (4.1) for the triplet  $A, B, C$  is minimal if and only if the two matrices given in (3.9) both have rank  $p$ , where we recall that the size of  $A$  is  $p \times p$ , that of  $B$  is  $p \times 1$ , and that of  $C$  is  $1 \times p$ . On the other hand, the ranks of the two matrices in (3.9) are both  $p$  if and only if (4.17) and (4.18) have only the trivial solutions  $g = 0$  and  $h = 0$ , respectively. ■

For any fixed  $x_0 \in \mathbf{R}$ , by shifting the dummy integration variable in (4.9) we get

$$N(x) = e^{-A(x-x_0)}N(x_0)e^{-A^\dagger(x-x_0)},$$

and similarly from (4.8) for any  $x_0, t_0 \in \mathbf{R}$  we get

$$Q(x; t) = e^{-A^\dagger(x-x_0)+4i(A^\dagger)^2(t-t_0)}Q(x_0; t_0)e^{-A(x-x_0)-4iA^2(t-t_0)}.$$

Thus, we have the following observations.

**Corollary 4.5** *Assume that the eigenvalues of  $A$  have positive real parts. Then, the matrix  $N(x)$  defined in (4.9) is invertible for all  $x \in \mathbf{R}$  if and only if it is invertible at any one particular value of  $x$ . Similarly,  $Q(x; t)$  defined in (4.8) is invertible for all  $x, t \in \mathbf{R}$  if and only if it is invertible at any one particular point on the  $xt$ -plane.*

**Proposition 4.6** *If the eigenvalues of  $A$  have positive real parts, then the matrix  $\Gamma(x; t)$  defined in (4.7) satisfies  $\Gamma(x; t) \rightarrow I$  as  $x \rightarrow +\infty$ . Additionally, if  $Q(0; 0)$  and  $N(0)$  given in (4.14) are invertible, then  $\Gamma(x; t)^{-1} \rightarrow 0$  exponentially as  $x \rightarrow -\infty$ , where  $I$  and  $0$  are the  $p \times p$  unit and zero matrices, respectively.*

PROOF: As stated in Proposition 4.1, since the integrals in (4.8) and (4.9) converge,  $\Gamma(x; t) \rightarrow I$  as  $x \rightarrow +\infty$  follows from (4.7)-(4.9). To obtain the limit for  $\Gamma(x; t)^{-1}$  as  $x \rightarrow -\infty$ , let us first define

$$Y(x; t) := e^{A^\dagger x} \Gamma(x; t) e^{A^\dagger x}. \quad (4.19)$$

Using (4.13) in (4.19) we get

$$Y(x; t) = Q(0; t) e^{-2Ax} N(0) \left[ I + N(0)^{-1} e^{2Ax} Q(0; t)^{-1} e^{2A^\dagger x} \right]. \quad (4.20)$$

Note that, from Theorem 4.2 it follows that  $N(0)^{-1}$  and  $e^{2Ax} Q(0; t)^{-1} e^{2A^\dagger x}$  are positive selfadjoint matrices. Using the Sherman-Morrison-Woodbury formula [23] as in the proof of Theorem 4.2, we see that the inverse of the matrix in the brackets in (4.20) exists, and for all  $x \in \mathbf{R}$  we have

$$Y(x; t)^{-1} = \left[ I + N(0)^{-1} e^{2Ax} Q(0; t)^{-1} e^{2A^\dagger x} \right]^{-1} N(0)^{-1} e^{2Ax} Q(0; t)^{-1}. \quad (4.21)$$

Further, since the eigenvalues of  $A$  and  $A^\dagger$  have strictly positive real parts, for each fixed  $t \in \mathbf{R}$  we conclude, as in the proof of Proposition 4.1, that there exists  $\epsilon > 0$  such that  $\|e^{Ax}\| = O(e^{\epsilon x})$  and  $\|e^{A^\dagger x}\| = O(e^{\epsilon x})$  as  $x \rightarrow -\infty$  in any matrix norm  $\|\cdot\|$ . Hence, from



(4.21) we see that  $Y(x; t)^{-1} \rightarrow 0$  exponentially as  $x \rightarrow -\infty$ , and writing (4.19) in the form

$$\Gamma(x; t)^{-1} = e^{A^\dagger x} Y(x; t)^{-1} e^{A^\dagger x},$$

we also see that  $\Gamma(x; t)^{-1} \rightarrow 0$  exponentially as  $x \rightarrow -\infty$ . ■

## 5. FURTHER PROPERTIES OF OUR EXPLICIT SOLUTIONS

We have obtained certain explicit solutions to (1.1) in the form of (4.11) by starting with some rational scattering data for (2.1) and by constructing the corresponding matrices  $A$ ,  $B$ , and  $C$  given in (3.5)-(3.7), respectively. In this section we will show that (4.11) is a solution to (1.1) no matter how the triplet  $A$ ,  $B$ ,  $C$  is chosen, as long as the matrix  $\Gamma(x; t)$  defined in (4.7) is invertible. For example, from Theorem 4.2 it follows that  $\Gamma(x; t)^{-1}$  exists on the entire  $xt$ -plane and thus (4.11) is a solution to (1.1) when the eigenvalues of  $A$  have positive real parts.

The purpose of this section is threefold. We will first obtain some useful representations for  $|u(x, t)|^2$  corresponding to  $u(x, t)$  given in (4.11). Next, we will prove that  $u(x, t)$  given in (4.11) is a solution to (1.1) as long as  $\Gamma(x; t)^{-1}$  exists. Then, we will consider further properties of such solutions.

We can evaluate  $|u(x, t)|^2$  from (4.11) directly. Alternatively, we can recover it by using the time-evolved analog of (2.10), namely

$$\int_x^\infty dz |u(z, t)|^2 = -2 G(x, x; t), \quad |u(x, t)|^2 = 2 \frac{\partial G(x, x; t)}{\partial x}, \quad (5.1)$$

where, in comparison with (2.8), we see that

$$G(x, y; t) := - \int_x^\infty dz \Omega(y + z; t)^\dagger K(x, z; t)^\dagger. \quad (5.2)$$

From (4.3), (4.8), (4.10), and (5.2), we get

$$G(x, y; t) = -B^\dagger e^{-A^\dagger y} \Gamma(x; t)^{-1} Q(x; t) e^{-Ax} B. \quad (5.3)$$

Using (5.3) in (5.1), with the help of (4.15), (4.16), and

$$(\Gamma^{-1})_x = -\Gamma^{-1}\Gamma_x\Gamma^{-1}, \quad (\Gamma^{-1})_t = -\Gamma^{-1}\Gamma_t\Gamma^{-1}, \quad (5.4)$$

we obtain

$$|u(x, t)|^2 = 4B^\dagger e^{-A^\dagger x} \Gamma(x; t)^{-1} [A^\dagger Q(x; t) + Q(x; t)A] [\Gamma(x; t)^\dagger]^{-1} e^{-Ax} B. \quad (5.5)$$

Next we show that  $|u(x, t)|^2$  can be expressed in a simple form in terms of the matrix  $\Gamma(x; t)$  defined in (4.7). As indicated in Theorem 4.2, recall that  $\Gamma(x; t)$  has a positive determinant for all  $x, t \in \mathbf{R}$  when the real parts of the eigenvalues of  $A$  are positive.

**Theorem 5.1** *The absolute square  $|u(x, t)|^2$  of the solution to the NLS equation can be written directly in terms of the determinant of the matrix  $\Gamma(x; t)$  defined in (4.7) so that*

$$|u(x, t)|^2 = \frac{\partial}{\partial x} \left[ \frac{\partial \det \Gamma(x; t) / \partial x}{\det \Gamma(x; t)} \right] = \frac{\partial^2}{\partial x^2} [\log (\det \Gamma(x; t))]. \quad (5.6)$$

PROOF: In terms of a matrix trace, from (5.1) and (5.3) we get

$$|u(x, t)|^2 = -2 \left[ B^\dagger e^{-A^\dagger x} \Gamma^{-1} Q e^{-Ax} B \right]_x = 2 \text{tr} [\Gamma^{-1} Q N_x]_x, \quad (5.7)$$

where we have used (4.9) and the fact that in evaluating the trace of a product of two matrices the order in the product can be changed. With the help of (4.7), (4.15), (4.16), and the trace properties we obtain

$$\text{tr} [\Gamma^{-1} Q N_x] = \text{tr} [-A - A^\dagger + (\Gamma^\dagger)^{-1} A + \Gamma^{-1} A^\dagger], \quad (5.8)$$

$$\text{tr} [\Gamma^{-1} Q_x N] = \text{tr} [-A - A^\dagger + (\Gamma^\dagger)^{-1} A + \Gamma^{-1} A^\dagger]. \quad (5.9)$$

Thus, from (5.7)-(5.9) with the help of (4.7) we get

$$2 \text{tr} [\Gamma^{-1} Q N_x] = \text{tr} [\Gamma^{-1} Q_x N + \Gamma^{-1} Q N_x] = \text{tr} [\Gamma^{-1} \Gamma_x],$$

and hence

$$|u(x, t)|^2 = \text{tr} [\Gamma^{-1} \Gamma_x]_x,$$

which can also be written as (5.6), as indicated in Theorem 7.3 on p. 38 of [18]. ■

We remark that (5.6) is a generalization of the formula given at the end of Section 3 of [37], where the formula was obtained for the  $n$ -soliton solution with simple poles. Thus, our method handles the bound states with nonsimple poles easily even though nonsimple poles have always caused complications in other methods and have mostly been avoided in the literature.

Let us also remark that (1.1) has infinitely many conserved quantities expressed as trace formulas. One such trace formula is given in the following.

**Proposition 5.2** *When the eigenvalues of the matrix  $A$  have positive real parts, the function  $u(x, t)$  given in (4.11) satisfies the trace formula*

$$\int_{-\infty}^{\infty} dx |u(x, t)|^2 = \text{tr} [A + A^\dagger] = 2 \sum_{j=1}^{m+n} n_j \text{Im}[\lambda_j], \quad (5.10)$$

where  $\lambda_j$  and  $n_j$  are the poles in  $\mathbf{C}^+$  and the corresponding multiplicities, respectively, as in (3.3).

PROOF: From (5.7) and (5.8) we see that

$$\int_{-\infty}^{\infty} dx |u(x, t)|^2 = \text{tr} [-A - A^\dagger + (\Gamma^\dagger)^{-1} A + \Gamma^{-1} A^\dagger] \Big|_{-\infty}^{\infty}.$$

As indicated in Proposition 4.6, we have  $\Gamma(x; t) \rightarrow I$  as  $x \rightarrow +\infty$  and  $\Gamma(x; t)^{-1} \rightarrow 0$  as  $x \rightarrow -\infty$ . Thus, we get the first equality in (5.10). Using (3.3) and (3.5), we can write the trace of  $(A + A^\dagger)$  in terms of the multiplicities and imaginary parts of  $\lambda_j$  as indicated in the second equality in (5.10). ■

**Theorem 5.3** *The function  $u(x, t)$  given in (4.11) satisfies (1.1) with any  $p \times p$  matrix  $A$ , column  $p$ -vector  $B$ , and row  $p$ -vector  $C$  as long as the matrix  $\Gamma(x; t)$  defined in (4.7) is*

invertible. In particular, if all eigenvalues of  $A$  have positive real parts, then  $u(x, t)$  given in (4.11) satisfies (1.1) on the entire  $xt$ -plane.

PROOF: With the help of (4.15), (4.16), and (5.4), through straightforward differentiation and after some simplifications, from (4.11) we get

$$iu_t = 8B^\dagger e^{-A^\dagger x} \Gamma^{-1} [(A^\dagger)^2 + QA^2N] \Gamma^{-1} e^{-A^\dagger x + 4i(A^\dagger)^2 t} C^\dagger, \quad (5.11)$$

$$\begin{aligned} u_x &= 4B^\dagger e^{-A^\dagger x} \Gamma^{-1} [A^\dagger - QAN] \Gamma^{-1} e^{-A^\dagger x + 4i(A^\dagger)^2 t} C^\dagger, \\ u_{xx} &= 8B^\dagger e^{-A^\dagger x} \Gamma^{-1} [(A^\dagger)^2 - 2QAN\Gamma^{-1}QAN + 2A^\dagger\Gamma^{-1}QAN - 2A^\dagger\Gamma^{-1}A^\dagger \\ &\quad + 2QAN\Gamma^{-1}A^\dagger + QA^2N] \Gamma^{-1} e^{-A^\dagger x + 4i(A^\dagger)^2 t} C^\dagger, \end{aligned} \quad (5.12)$$

$$2uu^\dagger u = -16B^\dagger e^{-A^\dagger x} \Gamma^{-1} [(A^\dagger Q + QA)(\Gamma^\dagger)^{-1}(AN + NA^\dagger)] \Gamma^{-1} e^{-A^\dagger x + 4i(A^\dagger)^2 t} C^\dagger. \quad (5.13)$$

Using (4.16) and (5.11)-(5.13), and noting that  $u^\dagger = \bar{u}$ , we verify that (1.1) is satisfied. Let us note that (5.13) could also be obtained directly by multiplying (4.11) and (5.5). ■

**Theorem 5.4** *Assume that the eigenvalues of  $A$  have positive real parts and that the matrices  $Q(0; 0)$  and  $N(0)$  given in (4.14) are invertible, or equivalently, assume that the representation in (3.8) with the triplet  $A, B, C$  is minimal and the eigenvalues of  $A$  have positive real parts. Then, for each fixed  $t \in \mathbf{R}$  the solution  $u(x, t)$  given in (4.11) vanishes exponentially as  $x \rightarrow \pm\infty$ .*

PROOF: From (4.11) and the fact that  $\Gamma(x; t) \rightarrow I$  as  $x \rightarrow +\infty$ , it follows that  $u(x, t) \rightarrow 0$  exponentially as  $x \rightarrow +\infty$  for each fixed  $t \in \mathbf{R}$ . Let us write (4.11) as

$$u(x, t) = -2B^\dagger Y(x; t)^{-1} e^{4i(A^\dagger)^2 t} C^\dagger, \quad (5.14)$$

where  $Y(x; t)$  is the matrix defined in (4.19). In the proof of Proposition 4.6, we have shown that  $Y(x; t)^{-1} \rightarrow 0$  exponentially as  $x \rightarrow -\infty$ . Hence, from (5.14) we can conclude that for each fixed  $t \in \mathbf{R}$  we have  $u(x, t) \rightarrow 0$  exponentially as  $x \rightarrow -\infty$ . ■

Let us remark that, if the eigenvalues of  $A$  have positive real parts, when extended to the entire  $x$ -axis the solutions given in (4.11) become multisoliton solutions, where the

number of solitons, multiplicity of the corresponding poles, and norming constants can be chosen at will. This can also be seen by analytically continuing the time-evolved Jost solution  $\psi(\lambda, x; t)$  to the entire  $x$ -axis, by using (2.3), (2.9), and

$$\frac{L(\lambda; t)}{T(\lambda; t)} = \lim_{x \rightarrow -\infty} \int_x^\infty dy K(x, y; t) e^{i\lambda(y-x)}, \quad (5.15)$$

by evaluating the integral with help of (4.10), and by observing that the limit in (5.15) vanishes.

## 6. GENERALIZATION

In some parts of Sections 3-5 we have assumed that  $\lambda_j$  values appearing in (3.3) and in the matrix  $A$  given in (3.5) are all located in  $\mathbf{C}^+$ . In this section we relax that restriction and allow some or all  $\lambda_j$  to be located in the lower half complex plane  $\mathbf{C}^-$ . Our only restriction will be that no  $\lambda_j$  will be real and no two distinct  $\lambda_j$  will be symmetrically located with respect to the real axis in the complex plane. This restriction is mathematically equivalent to the disjointness of the sets  $\{\lambda_j\}_{j=1}^{m+n}$  and  $\{\bar{\lambda}_j\}_{j=1}^{m+n}$ . Under this restriction we will show that  $u(x, t)$  given in (4.11) is a solution to (1.1) in any region on the  $xt$ -plane in which the matrix  $\Gamma(x; t)$  defined in (4.7) is invertible. The only change we need is that  $Q(x; t)$  and  $N(x)$  will no longer be defined as in (4.8) and (4.9), but instead they will be given as in (4.13), where we now let

$$Q(0; 0) = \frac{1}{2\pi} \int_\gamma d\lambda (\lambda + iA^\dagger)^{-1} C^\dagger C (\lambda - iA)^{-1}, \quad (6.1)$$

$$N(0) = \frac{1}{2\pi} \int_\gamma d\lambda (\lambda - iA)^{-1} B B^\dagger (\lambda + iA^\dagger)^{-1}, \quad (6.2)$$

with  $\gamma$  being any positively oriented simple closed contour enclosing all  $\lambda_j$  in such a way that all  $\bar{\lambda}_j$  lie outside  $\gamma$ .

As the following proposition shows, the quantities given in (6.1) and (6.2) are the unique (selfadjoint) solutions to the respective Lyapunov equations

$$Q(0; 0) A + A^\dagger Q(0; 0) = C^\dagger C, \quad (6.3)$$

$$A N(0) + N(0) A^\dagger = B B^\dagger. \quad (6.4)$$

We note that, using (4.13), we could also write (6.3) and (6.4) in the equivalent form

$$Q(x; t) A + A^\dagger Q(x; t) = e^{-A^\dagger x + 4i(A^\dagger)^2 t} C^\dagger C e^{-Ax - 4iA^2 t}, \quad (6.5)$$

$$A N(x) + N(x) A^\dagger = e^{-Ax} B B^\dagger e^{-A^\dagger x}. \quad (6.6)$$

**Proposition 6.1** *Assume that none of the eigenvalues of  $A$  are purely imaginary and that no two eigenvalues of  $A$  are symmetrically located with respect to the imaginary axis. Equivalently, assume that  $\{\lambda_j\}_{j=1}^{m+n}$  and  $\{\bar{\lambda}_j\}_{j=1}^{m+n}$  are disjoint, where  $\lambda_j$  are the complex constants appearing in (3.3) and (3.5). We then have the following:*

- (i) *The matrix equations given in (6.3) and (6.4) are each uniquely solvable.*
- (ii) *The unique solutions  $Q(0; 0)$  and  $N(0)$  are selfadjoint matrices.*
- (iii) *The unique solutions are given by (6.1) and (6.2), respectively.*

PROOF: Note that (i) and (iii) directly follow from Theorem 4.1 in Section I.4 of [21]. It is straightforward to show that the adjoint of any solution to (6.3) or (6.4) is also a solution to the same equation, and hence the unique solutions  $Q(0; 0)$  and  $N(0)$  must be selfadjoint. ■

Next, without requiring that all  $\lambda_j$  appearing in (3.5) be located in  $\mathbf{C}^+$ , we will prove that the matrix  $u(x, t)$  given in (4.11) is a solution to (1.1) as long as  $\Gamma(x; t)$  defined in (4.7) is invertible. First, we will write (4.11) in a slightly different but equivalent form. Define

$$\Lambda(x; t) := I + P(x; t)^\dagger Q(0; 0) P(x; t) N(0), \quad P(x; t) := e^{-2Ax - 4iA^2 t}. \quad (6.7)$$

Note that  $\Gamma(x; t)$  is invertible if and only if  $\Lambda(x; t)$  is invertible because, by using (4.7), (4.13), and (6.7), we see that

$$\Gamma(x; t) = e^{A^\dagger x} \Lambda(x; t) e^{-A^\dagger x}. \quad (6.8)$$

With the help of (6.8) we can write (4.11) in the equivalent form

$$u(x, t) = -2B^\dagger \Lambda(x; t)^{-1} P(x; t)^\dagger C^\dagger. \quad (6.9)$$

**Theorem 6.2** *Assume that none of the eigenvalues of the matrix  $A$  in (3.5) are purely imaginary and that no two eigenvalues of  $A$  are symmetrically located with respect to the imaginary axis. Equivalently, assume that  $\{\lambda_j\}_{j=1}^{m+n}$  and  $\{\bar{\lambda}_j\}_{j=1}^{m+n}$  are disjoint, where  $\lambda_j$  are the complex constants appearing in (3.3) and (3.5). Then, the quantity  $u(x, t)$  given in (4.11), or equivalently in any of (4.12), (5.14), and (6.9) is a solution to (1.1) in any region of the  $xt$ -plane where the matrix  $\Lambda(x; t)$  defined in (6.7) or equivalently the matrix  $\Gamma(x; t)$  given in (4.7) is invertible.*

PROOF: In our proof let us write  $u, \Lambda, P, Q, N$  for  $u(x, t), \Lambda(x; t), P(x; t), Q(0; 0), N(0)$ , respectively. Without explicitly mentioning it, we will use the selfadjointness  $Q^\dagger = Q$  and  $N^\dagger = N$  established in Proposition 6.1 as well as the fact that  $P$  is invertible. Proceeding as in the proof of Theorem 5.3, using straightforward differentiation on (6.9) and after some simplification we obtain

$$iu_t = 8B^\dagger \Lambda^{-1} [(A^\dagger)^2 + P^\dagger Q A^2 P N] \Lambda^{-1} P^\dagger C^\dagger, \quad (6.10)$$

where we have used the fact that

$$\Lambda = I + P^\dagger Q P N, \quad (\Lambda^{-1})_t = -\Lambda^{-1} \Lambda_t \Lambda^{-1}, \quad P_t = -4iA^2 P, \quad AP = PA. \quad (6.11)$$

Similarly, by using (6.11) and

$$P_x = -2AP, \quad (\Lambda^{-1})_x = -\Lambda^{-1} \Lambda_x \Lambda^{-1},$$

after some simplifications we obtain

$$u_x = 4B^\dagger \Lambda^{-1} [A^\dagger - P^\dagger Q A P N] \Lambda^{-1} P^\dagger C^\dagger,$$

$$\begin{aligned}
u_{xx} = 8B^\dagger \Lambda^{-1} [(A^\dagger)^2 - 2A^\dagger \Lambda^{-1} A^\dagger + P^\dagger Q A^2 P N + 2A^\dagger \Lambda^{-1} P^\dagger Q A P N \\
+ 2P^\dagger Q A P N \Lambda^{-1} A^\dagger - 2P^\dagger Q A P N \Lambda^{-1} P^\dagger Q A P N] \Lambda^{-1} P^\dagger C^\dagger.
\end{aligned} \tag{6.12}$$

Next, with the help of (6.3) and (6.4) and using  $|u|^2 u = uu^\dagger u$ , we obtain

$$\begin{aligned}
2|u|^2 u = -16B^\dagger \Lambda^{-1} [P^\dagger Q A P (\Lambda^\dagger)^{-1} A N + P^\dagger Q A P (\Lambda^\dagger)^{-1} N A^\dagger \\
+ P^\dagger A^\dagger Q P (\Lambda^\dagger)^{-1} A N + P^\dagger A^\dagger Q P (\Lambda^\dagger)^{-1} N A^\dagger] \Lambda^{-1} P^\dagger C^\dagger.
\end{aligned} \tag{6.13}$$

We see that (1.1) is satisfied, which is verified by adding (6.10), (6.12), and (6.13) side by side and by using

$$QPN = (P^\dagger)^{-1}(\Lambda - I), \quad (\Lambda^\dagger)^{-1}N = N\Lambda^{-1}, \quad NP^\dagger Q = (\Lambda^\dagger - I)P^{-1},$$

which directly follows from (6.7) and the selfadjointness of  $Q$  and  $N$ . ■

As the next theorem shows, if we remove the restriction  $\lambda_j \in \mathbf{C}^+$  then the result in Theorem 5.1 still remains valid in any region in the  $xt$ -plane where  $\Gamma(x; t)$  or equivalently  $\Lambda(x; t)$  is invertible.

**Theorem 6.3** *Assume that none of the eigenvalues of the matrix  $A$  in (3.5) are purely imaginary and that no two eigenvalues of  $A$  are symmetrically located with respect to the imaginary axis. Equivalently, assume that  $\{\lambda_j\}_{j=1}^{m+n}$  and  $\{\bar{\lambda}_j\}_{j=1}^{m+n}$  are disjoint, where  $\lambda_j$  are the complex constants appearing in (3.3) and (3.5). Then, in any region of the  $xt$ -plane where the matrix  $\Lambda(x; t)$  defined in (6.7) or equivalently the matrix  $\Gamma(x; t)$  given in (4.7) is invertible, the solution  $u(x, t)$  given in (4.11) or equivalently in (6.9) satisfies (5.6) or equivalently*

$$|u(x, t)|^2 = \text{tr} \left[ \frac{\partial}{\partial x} \left( \Gamma(x, t)^{-1} \frac{\partial \Gamma(x; t)}{\partial x} \right) \right] = \frac{\partial}{\partial x} \left[ \frac{\partial \det \Gamma(x; t) / \partial x}{\det \Gamma(x; t)} \right], \tag{6.14}$$

$$|u(x, t)|^2 = \text{tr} \left[ \frac{\partial}{\partial x} \left( \Lambda(x, t)^{-1} \frac{\partial \Lambda(x; t)}{\partial x} \right) \right] = \frac{\partial}{\partial x} \left[ \frac{\partial \det \Lambda(x; t) / \partial x}{\det \Lambda(x; t)} \right]. \tag{6.15}$$

PROOF: Let us write  $u, \Lambda, P, Q, N$  for  $u(x, t), \Lambda(x; t), P(x; t), Q(x; t), N(x)$ , respectively. Using the fact that, in evaluating the trace of a product of two matrices we can change the order in the matrix product, from (6.8) we obtain

$$\text{tr} [\Gamma^{-1} \Gamma_x] = \text{tr} [\Lambda^{-1} \Lambda_x],$$



and hence it is sufficient to prove only (6.14). From (4.13) it follows that (6.5) and (6.6) are equivalent to the first two equations, respectively, in (4.15). Note that (4.16) is still valid and is a direct consequence of (4.7) and the selfadjointness of  $Q$  and  $N$ . Proceeding as in the proof of Theorem 5.1, with the help of (4.15), (4.16), and (5.4) we obtain

$$\begin{aligned}\operatorname{tr} [\Gamma^{-1}\Gamma_x] &= 2\operatorname{tr} [-A - A^\dagger + (\Gamma^\dagger)^{-1}A + \Gamma^{-1}A^\dagger], \\ \operatorname{tr} [\Gamma^{-1}\Gamma_x]_x &= 4\operatorname{tr} [\Gamma^{-1}(A^\dagger)^2 + (\Gamma^\dagger)^{-1}A^2 - \Gamma^{-1}A^\dagger\Gamma^{-1}A^\dagger \\ &\quad - (\Gamma^\dagger)^{-1}A(\Gamma^\dagger)^{-1}A + 2\Gamma^{-1}QAN\Gamma^{-1}A^\dagger].\end{aligned}\tag{6.16}$$

On the other hand, using the fact that  $|u|^2 = uu^\dagger$ , from (4.11) we obtain

$$|u|^2 = 4\operatorname{tr} [(AN + NA^\dagger)\Gamma^{-1}(QA + A^\dagger Q)(\Gamma^\dagger)^{-1}],\tag{6.17}$$

where we have also used (6.5) and (6.6). Using (4.16) and the aforementioned property of the matrix trace, we can simplify the right hand side of (6.17) and show that it is equal to the right hand side of (6.16). Finally, as indicated in the proof of Theorem 5.1, the second equalities in (6.14) and (6.15) follow from Theorem 7.3 on p. 38 of [18]. ■

## 7. EXAMPLES

Specific examples of our exact solutions can be obtained from the explicit formula (4.11), or equivalently from any one of (4.12), (5.14), and (6.9), by specifying  $A$ ,  $B$ , and  $C$ , where  $\Gamma(x; t)$  is the matrix defined in (4.7). We have made available various Mathematica notebooks [39] in which the user can easily perform the following steps and display the corresponding exact solution  $u(x, t)$  explicitly in terms of exponential, trigonometric, and polynomial functions, verify that the resulting  $u(x, t)$  satisfies (1.1), and animate  $|u(x, t)|$ .

- (i) Input the matrices  $A$ ,  $B$ ,  $C$ .
- (ii) Evaluate the matrix  $\Gamma(x; t)$  as in (4.7), where  $Q(x; t)$  and  $N(x)$  are the matrices appearing in (4.13). In case all the eigenvalues of  $A$  lie in the right half complex plane, evaluate  $Q(0; 0)$  and  $N(0)$  explicitly as in (4.14) with the help of `MatrixExp`,

which is used to evaluate matrix exponentials in Mathematica. In case some or all eigenvalues of  $A$  lie in the left half complex plane, use (6.1) and (6.2) instead in order to evaluate explicitly  $Q(0;0)$  and  $N(0)$ , respectively.

- (iii) Having obtained  $\Gamma(x;t)$ , use (4.11) or one of its equivalents (4.12), (5.14), and (6.9) to display  $u(x,t)$  explicitly in terms of exponential, trigonometric, and polynomial functions.
- (iv) Using (5.6) or (4.11), evaluate  $|u(x,t)|^2$  exactly and animate  $|u(x,t)|$ .
- (v) As an option, evaluate the quantities  $iu_t$ ,  $u_{xx}$ , and  $2|u|^2u$ , and verify directly that (1.1) is satisfied.

**Example 7.1** The well-known “ $n$ -soliton” to (1.1) is obtained when  $R(\lambda) \equiv 0$  and  $T(\lambda)$  has  $n$  simple bound-state poles in  $\mathbf{C}^+$ . In this case, from (3.5)-(3.7) we see that  $A$ ,  $B$ , and  $C$  are given by

$$A = \begin{bmatrix} -i\lambda_1 & 0 & \dots & 0 \\ 0 & -i\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -i\lambda_n \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad C = [c_1 \quad c_2 \quad \dots \quad c_n], \quad (7.1)$$

where  $\lambda_j$  are distinct and all lie in  $\mathbf{C}^+$ . Using (4.7)-(4.9), the  $(\alpha, \beta)$ -entries of the matrices  $Q(x;t)$ ,  $N(x)$ , and  $\Gamma(x;t)$  are easily evaluated as

$$N_{\alpha\beta} = \frac{ie^{i(\lambda_\alpha - \bar{\lambda}_\beta)x}}{\lambda_\alpha - \bar{\lambda}_\beta}, \quad Q_{\alpha\beta} = \frac{i\bar{c}_\alpha c_\beta e^{i(\lambda_\beta - \bar{\lambda}_\alpha)x + 4i(\lambda_\beta^2 - \bar{\lambda}_\alpha^2)t}}{\lambda_\beta - \bar{\lambda}_\alpha},$$

$$\Gamma_{\alpha\beta} = \delta_{\alpha\beta} - \sum_{\gamma=1}^n \frac{\bar{c}_\alpha c_\gamma e^{i(2\lambda_\gamma - \bar{\lambda}_\alpha - \bar{\lambda}_\beta)x + 4i(\lambda_\gamma^2 - \bar{\lambda}_\alpha^2)t}}{(\lambda_\gamma - \bar{\lambda}_\alpha)(\lambda_\gamma - \bar{\lambda}_\beta)},$$

where  $\delta_{\alpha\beta}$  is the Kronecker delta. A Mathematica notebook [39] is available, where the user can specify  $n$  and  $\{\lambda_j, c_j\}_{j=1}^n$  and display the corresponding  $u(x,t)$  explicitly in terms of exponential, trigonometric, and polynomial functions and animate  $|u(x,t)|$ .

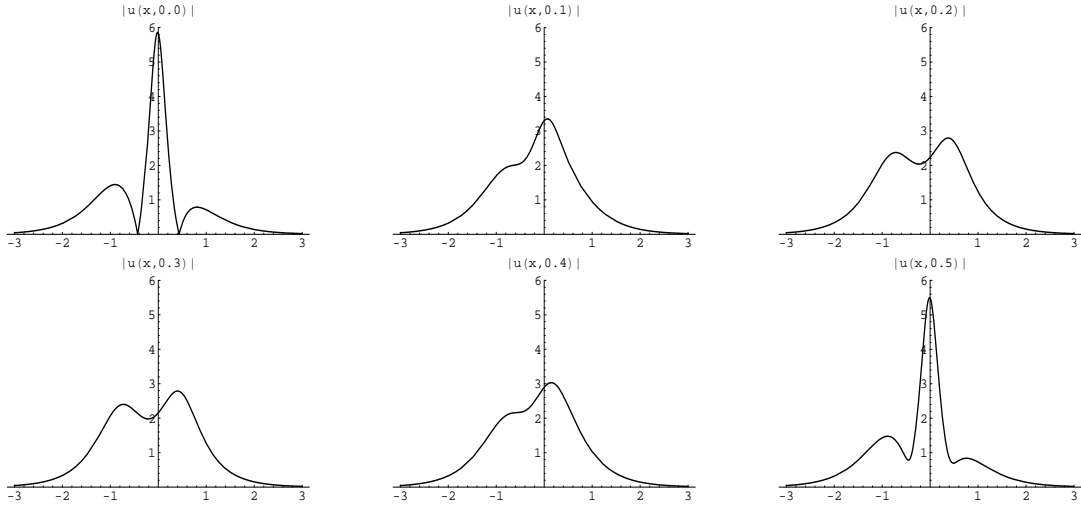
**Example 7.2** Choosing

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = [1 \quad -1],$$

we evaluate  $Q(0;0)$  and  $N(0)$  using (6.3) and (6.4), respectively. Then, with the help of (6.7) and (6.9) we obtain

$$u(x, t) = \frac{8e^{4it}(9e^{-4x} + 16e^{4x}) - 32e^{16it}(4e^{-2x} + 9e^{2x})}{-128 \cos(12t) + 4e^{-6x} + 16e^{6x} + 81e^{-2x} + 64e^{2x}}. \quad (7.2)$$

Note that one of the eigenvalues of  $A$  in this example is negative and the solution in (7.2) is not a soliton solution. A Mathematica notebook containing the animation of (7.2) is available [39].



**Fig. 7.1** Snapshots of  $|u(x, t)|$  of Example 7.2 at  $t = 0.0, 0.1, 0.2, 0.3, 0.4,$  and  $0.5$ .

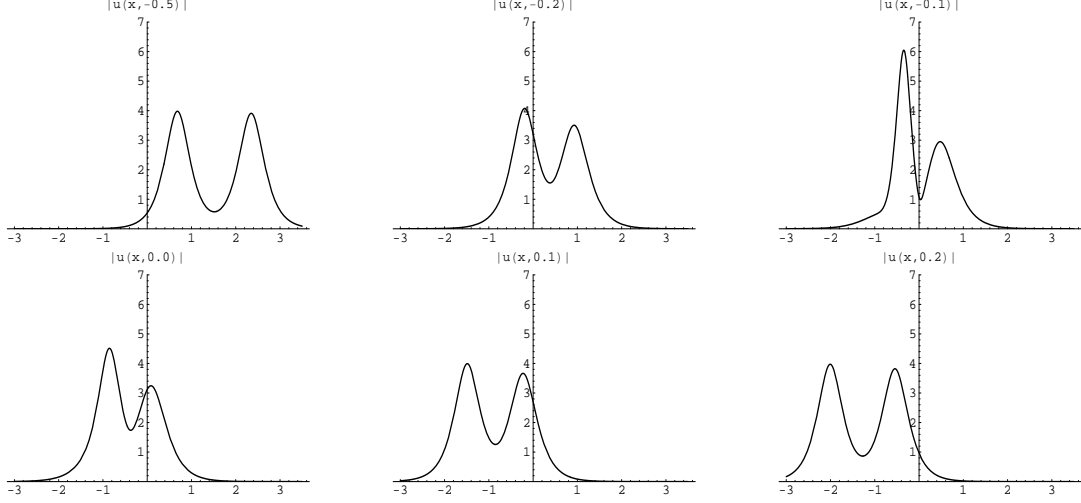
**Example 7.3** Choosing

$$A = \begin{bmatrix} 2 - i & -1 \\ 0 & 2 - i \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 + 2i \quad -1 + 4i],$$

we get  $u(x, t) = \text{num}(x, t)/\text{den}(x, t)$ , where

$$\begin{aligned} \text{num}(x, t) &:= 1024e^{4(x+4t)-2i(x-6t)} [(12 - 9i) + 100t + (5 - 10i)x] \\ &\quad + 131072e^{12(x+4t)-2i(x-6t)} [(1 + 4i) + (24 + 32i)t - (2 - 4i)x], \\ \text{den}(x, t) &:= 25 + 65536e^{16(4t+x)} \\ &\quad + 512e^{8(4t+x)} [12800t^2 + 64(20x + 43)t + 160x^2 + 304x + 207]. \end{aligned}$$

The solution in this example can be described as a soliton of double multiplicity, and its Mathematica animation is available [39].



**Fig. 7.2** Snapshots of  $|u(x, t)|$  of Example 7.3 at  $t = -0.5, -0.2, -0.1, 0.0, 0.1,$  and  $0.2$ .

**Example 7.4** Choosing

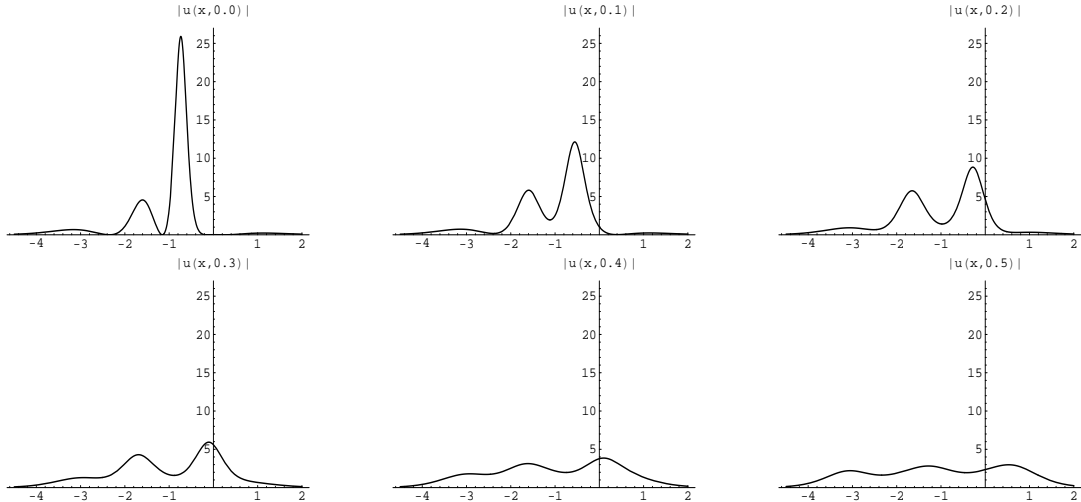
$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0],$$

we easily obtain  $u(x, t) = \text{num}(x, t)/\text{den}(x, t)$ , where

$$\begin{aligned} \text{num}(x, t) := & 32e^{-2(x-2it)} \{[-32768x^2 + 524288t^2 + 262144itx - 65536it] \\ & + e^{-4x} [90112t^2 + 15872x^2 + 131072t^2x^2 + 4096x^4 + 196608xt^2 \\ & + 12288x^3 + 9216x + 1344 + 1048576t^4 - 32768itx^2 - 35840it \\ & - 61440itx] + e^{-8x} [128t^2 - 8x^2 - 24x - 15 - 112it - 64itx]\}, \end{aligned}$$

$$\begin{aligned} \text{den}(x, t) := & 262144 + e^{-4x} [262144x^4 + 589824x^2 + 393216x + 524288x^3 \\ & + 67108864t^4 + 8388608x^2t^2 + 122880] + e^{-8x} [16384x^3 + 4096x^4 \\ & + 1048576t^4 + 15360x + 344064t^2 + 24576x^2 + 131072x^2t^2 \\ & + 393216xt^2 + 3648] + e^{-12x}. \end{aligned}$$

The solution in this example can be described as a soliton of triple multiplicity. A Mathematica notebook [39] is available for this example and the corresponding animation.



**Fig. 7.3** Snapshots of  $|u(x, t)|$  of Example 7.4 at  $t = 0.0, 0.1, 0.2, 0.3, 0.4,$  and  $0.5$ .

In Figures 7.1-7.3 we present some snapshots of  $|u(x, t)|$  appearing in Examples 7.2-7.4. Further examples of exact solutions to (1.1) expressed in terms of exponential, trigonometric, and polynomial functions as well as their animations can be obtained with the help of available Mathematica notebooks [39]. It can be directly verified that  $u(x, t)$  given in the above examples all satisfy (1.1). When the matrix size for the  $A, B, C$  becomes large, such expressions become lengthy and yet can easily be displayed with the help of Mathematica or any other symbolic software.

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