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FINITE PROJECTIVE DIMENSION AND THE VANISHING OF $\operatorname{Ext}_R(M, M)$

DAVID A. JORGENSEN

Abstract. We show that for a finitely generated module M over a complete intersection R, the vanishing of $\operatorname{Ext}^i_R(M, M)$ for a certain number of consecutive values of i starting at n forces the projective dimension of M to be at most $n - 1$. In particular, $\operatorname{Ext}^2_R(M, M) = 0$ if and only if the projective dimension of M over R is at most one. We also verify a conjecture of Auslander and Reiten for modules over commutative rings with certain typical behaviors, which augments the recent literature.

Introduction

In this paper we study the relationship between the vanishing of $\operatorname{Ext}^i_R(M, M)$ for various consecutive values of i, and the projective dimension of M. We do so when R is a commutative noetherian ring.

In Section One we extend a result of Jothilingam [13], and show that over a codimension c complete intersection R, $\operatorname{Ext}^i_R(M, M) = 0$ for $c+1$ consecutive values of i starting at n forces the projective dimension of M to be at most $n - 1$. We moreover show that $\operatorname{Ext}^2_R(M, M) = 0$ (a single Ext vanishing) forces $\operatorname{pd}_R M \leq 1$.

In Section Two we discuss a certain spectral sequence that allows one to relate Ext and Tor for modules over an arbitrary commutative ring.

In Section Three we use the spectral sequence from Section Two to verify for certain modules over commutative noetherian rings a conjecture of Auslander and Reiten [6] which proposes that the vanishing of $\operatorname{Ext}^i_R(M, M \oplus R)$ for all $i > 0$ characterizes projective modules. See [4, 1.9] for the commutative version dealt with below. While in some cases the validity of the conjecture is known, our proofs are new and give additional perspective to the conjecture.

1. Over complete intersections

Recall that a local noetherian ring $(R, \mathfrak{m})$ is a *complete intersection* if its completion in the $\mathfrak{m}$-adic topology is the quotient of a regular local ring Q by a Q-regular sequence $\boldsymbol{x} = x_1, \ldots, x_c$ contained in the maximal ideal of Q. If $\boldsymbol{x}$ is contained in the square of the maximal ideal of Q, then R has codimension c.

We say that a finitely generated R-module M is *Tor-rigid* if for all finitely generated R-modules N, $\operatorname{Tor}^R_i(M, N) = 0$ for some $i > 0$ implies that $\operatorname{Tor}^R_j(M, N) = 0$ for all $j \geq i$. The following is a generalization of a result of Jothilingam [13], his being the $c = 0$ case.

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Theorem 1.1. *Let M be a finitely generated Tor-rigid module over a commutative local ring R. Suppose $\boldsymbol{x}$ is an R-sequence of length $c \geq 0$ such that $(\boldsymbol{x})M = 0$. Then*

$$\operatorname{Ext}^i_{R/(\boldsymbol{x})}(M, M) = 0 \quad \textit{for } n \leq i \leq n + c$$

if and only if $\operatorname{pd}_{R/(\boldsymbol{x})} M < n$.

Proof. We induct on c. When $c = 0$ we have $(\boldsymbol{x}) = (0)$, and in this case the result is contained in the main theorem of [13]. Although that result was originally stated for regular local rings, over which all finitely generated modules are Tor-rigid [2],[16], the more general form stated here is what is actually proved in [13].

For $c > 0$, write $T = R/(x_1, \ldots, x_{c-1})$ and $S = R/(\boldsymbol{x})$. Then $S \cong T/(x_c)$, and the change of rings spectral sequence $\operatorname{Ext}^p_T(\operatorname{Tor}^S_q(T, M), M) \underset{p}{\Longrightarrow} \operatorname{Ext}^{p+q}_S(M, M)$ degenerates into a long exact sequence

$$\cdots \to \operatorname{Ext}^i_S(M, M) \to \operatorname{Ext}^i_T(M, M) \to \operatorname{Ext}^{i-1}_S(M, M) \to \operatorname{Ext}^{i+1}_S(M, M) \to \cdots$$

It follows that $\operatorname{Ext}^i_T(M, M) = 0$ for all $n + 1 \leq i \leq n + c$, and by induction we conclude that $\operatorname{pd}_T M < n + 1$. Thus M has finite CI-dimension over S (see [5] for the definition). Since $c > 0$, the index of at least one of the $\operatorname{Ext}^S_i(M, M)$ assumed to vanish in the hypothesis is even. Now Theorem 4.2 of [3] says that $\operatorname{pd}_S M < \infty$, and it follows that $\operatorname{pd}_S M = \operatorname{pd}_T M - 1 < n$. □

For complete intersections we get the following corollaries.

Corollary 1.2. *Suppose that R is a codimension c complete intersection, and M is a finitely generated R-module. Then*

$$\operatorname{Ext}^i_R(M, M) = 0 \quad \textit{for } n \leq i \leq n + c$$

if and only if $\operatorname{pd}_R M < n$.

Proof. We have $\widehat{R} = Q/(\boldsymbol{x})$ with Q a complete regular local ring, and $\boldsymbol{x}$ a length c Q-sequence contained in the square of the maximal ideal of Q. By faithful flatness of $R \hookrightarrow \widehat{R}$ we have

$$\operatorname{Ext}^i_{\widehat{R}}(\widehat{M}, \widehat{M}) = 0 \quad \text{for } n \leq i \leq n + c.$$

Since $\widehat{M}$ is Tor-rigid over Q, we have by 1.1, $\operatorname{pd}_R M = \operatorname{pd}_{\widehat{R}} \widehat{M} < n$. □

In special cases one can improve the results slightly:

Corollary 1.3. *Suppose that R is a codimension $c \geq 1$ complete intersection with $\widehat{R}$ the quotient of an unramified regular local ring, and let M be a finitely generated R-module. If $c = 1$ assume that either M has finite projective dimension, or finite length. Then*

$$\operatorname{Ext}^i_R(M, M) = 0 \quad \textit{for } n \leq i \leq n + c - 1$$

if and only if $\operatorname{pd}_R M < n$.

Proof. For $c = 1$, we have $\widehat{R} = Q/(x_1)$ with Q an unramified regular local ring. The $\widehat{R}$-module $\widehat{M}$ is Tor-rigid if either $\operatorname{pd}_{\widehat{R}} \widehat{M} < \infty$, by [16, Theorem 3], or if it has finite length, by [11, 2.4]. Thus the "$c = 0$" case of 1.1 gives the desired conclusion.

For $c > 1$, we have $\operatorname{Ext}^i_R(M, M) = 0$ for some i even. By [3, 4.2] M then has finite projective dimension over R, and therefore $\widehat{M}$ has the same over $\widehat{R}$, and so too over $Q/(x_1)$. Thus $\widehat{M}$ is Tor-rigid over $Q/(x_1)$, and the result follows from 1.1. □

Lifting. Suppose that R is the homomorphic image of a noetherian ring Q. Recall that a finitely generated R-module M is said to *lift* from R to Q if there exists a finitely generated Q-module M' such that $M \cong M' \otimes_Q R$ and $\operatorname{Tor}_i^Q(M', R) = 0$ for all $i > 0$. "What seems to be folklore to the deformation theorist," and proved in [4] is if Q is an algebra over a commutative complete local ring P with maximal ideal $\mathfrak{p}$ and R is the quotient of Q by a Q-regular sequence contained in $\mathfrak{p}$, then the non-triviality of classes in $\operatorname{Ext}^2_R(M, M)$ are the obstructions to lifting M to Q, and thus M has a lifting M' to Q whenever $\operatorname{Ext}^2_R(M, M) = 0$.

Due to the importance of the question of lifting, it seems worthwhile to have some information on the vanishing of $\operatorname{Ext}^2_R(M, M)$ in basic situations to which the conclusions of [4] apply. One such situation is when the ring R is a complete intersection. The following results show that here the vanishing of $\operatorname{Ext}^2_R(M, M)$ can only occur if M has projective dimension at most 1. Thus the condition of the entirety of the group $\operatorname{Ext}^2_R(M, M)$ vanishing in determining lifting in this situation is quite coarse, for liftable modules with projective dimension at least two are abundant (provided the dimension of R is at least two).

We need another extension of the result from [13].

Lemma 1.4. *Let M be a finitely generated Tor-rigid module over a commutative local ring R. Suppose $\boldsymbol{x}$ is both an R-sequence and an M-sequence of length $c \geq 0$. Then*

$$\operatorname{Ext}^n_{R/(\boldsymbol{x})}(M/(\boldsymbol{x})M, M/(\boldsymbol{x})M) = 0 \quad \textit{for some } n > 0$$

if and only if $\operatorname{pd}_{R/(\boldsymbol{x})} M/(\boldsymbol{x})M < n$.

Proof. By induction it suffices to consider the $c = 1$ case.

Since $x = x_1$ is regular on M, we have for any finitely generated $R/(x)$-module N the following standard isomorphisms

$$\operatorname{Tor}_i^{R/(x)}(M/xM, N) \cong \operatorname{Tor}_i^R(M, N) \quad \text{for all } i.$$

It follows that M/xM is Tor-rigid as an $R/(x)$-module. Hence 1.1 gives the desired conclusion. □

Proposition 1.5. *Let M be a finitely generated module over a complete intersection R. Then $\operatorname{Ext}^2_R(M, M) = 0$ if and only if $\operatorname{pd}_R M \leq 1$, and $\operatorname{Ext}^1_R(M, M) = \operatorname{Ext}^2_R(M, M) = 0$ if and only if M is free.*

Proof. We have $\widehat{R} = Q/(\boldsymbol{x})$ where Q is a complete regular local ring, and $\boldsymbol{x}$ is a Q-regular sequence contained in the square of the maximal ideal of Q. By the faithful flatness of $R \hookrightarrow \widehat{R}$, we have $\operatorname{Ext}^i_R(M, M) = 0$ if and only if $\operatorname{Ext}^i_{\widehat{R}}(\widehat{M}, \widehat{M}) = 0$, and $\operatorname{pd}_R M = \operatorname{pd}_{\widehat{R}} \widehat{M}$. Thus by hypothesis, $\operatorname{Ext}^2_{\widehat{R}}(\widehat{M}, \widehat{M}) = 0$. Now [4] shows that $\widehat{M}$ lifts to Q with lifting M', say. Then $\boldsymbol{x}$ is both a Q-regular sequence and an M'-regular sequence, and $M'/(\boldsymbol{x})M' \cong M$. Now 1.4 applies to show that $\operatorname{pd}_R M = \operatorname{pd}_{\widehat{R}} \widehat{M} < 2$. If moreover $\operatorname{Ext}^1_R(M, M) = 0$, then $\operatorname{pd}_R M = 0$, in other words, M is free. □

Remark 1.6. As is mentioned in the proof of 1.1, Theorem 4.2 of [3] shows that for a finitely generated module M over a complete intersection R, $\operatorname{Ext}^{2i}_R(M, M) = 0$ for some $i > 0$ implies that M has finite projective dimension over R. The following well-known example (see also [3, 4.3]) shows that the vanishing of $\operatorname{Ext}^{2i+1}_R(M, M)$ for some i has no such consequence. In particular, $\operatorname{Ext}^3_R(M, M) = 0$ does not in general imply that $\operatorname{pd}_R M < 3$, or even that $\operatorname{pd}_R M < \infty$: *let* $R = k[[X,Y]]/(XY)$, *where* k *is a field, and* $M = (x)$. *Then* $\operatorname{Ext}_R^{2i+1}(M,M) = 0$ *for all* $i > 0$, *and* $\operatorname{pd}_R M = \infty$.

We would like to know if over a complete intersection, $\operatorname{Ext}_R^4(M,M) = 0$ implies that M has projective dimension at most three. Indeed, we ask the following:

Question 1.7. *Let* M *be a finitely generated module of finite projective dimension over a complete intersection* R *of positive codimension. Does* $\operatorname{Ext}_R^n(M,M) = 0$ *imply that* $\operatorname{pd}_R M < n$*?*

Examining the proof of 1.1, a counterexample to this question would also yield a (rather special) counterexample to rigidity of Tor for a module of finite projective dimension over a complete intersection (see for example [8]).

It is shown in [14] that for $R = Q/(x)$ with x in the square of the maximal ideal of a regular local ring Q, and $\dim R \geq 2$, then there always exist finitely generated R-modules of finite projective dimension ≥ 2 which fail to lift to Q.

We summarize some of these remarks in the following diagram, for a finitely generated module M over a complete intersection $R = Q/(\boldsymbol{x})$, with Q a complete regular local ring and $\boldsymbol{x}$ a Q-regular sequence contained in the square of the maximal ideal of Q.

$$\operatorname{pd}_R M \leq 1 \iff \operatorname{Ext}^2(M,M) = 0$$
$$\Downarrow \quad \not\Uparrow$$
$$M \text{ lifts to } Q$$
$$\Downarrow \quad \not\Uparrow$$
$$\operatorname{pd}_R M < \infty \iff \operatorname{Ext}_R^{2i}(M,M) = 0 \text{ for some } i$$
$$\Downarrow \quad \not\Uparrow$$
$$\operatorname{Ext}_R^{2i+1}(M,M) = 0 \text{ for some } i$$

2. Relating Ext and Tor

In this section we discuss the spectral sequence associated to a natural first quadrant bicomplex which exploits the fact that over a commutative ring A, the cohomology modules $\operatorname{Ext}_A(X,Y)$ can be computed by taking homology after applying either $\operatorname{Hom}_A(-,Y)$ or $\operatorname{Hom}(-,A) \otimes_A Y$ to a free resolution of X. We make use of these results in the next section.

The 4-term exact sequences key to the proofs in [13] are from [1], and turn out to be the first four terms of the 5-term exact sequences from our spectral sequence. (See [9, Proposition 3.6] where the existence of such a spectral sequence is mentioned, but not explicitly given).

Let A be a commutative noetherian ring, X be a finitely generated A-module, and

$$F : \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to X \to 0$$

a projective resolution of X. We define for $n \geq 0$

$$\mathrm{D}^n X = \operatorname{coker}(F_n^* \xrightarrow{d_{n+1}^*} F_{n+1}^*).$$

Of course our definition of $\mathrm{D}^n X$ depends on the resolution chosen. However, when A is local and the resolution is chosen minimal, then $\mathrm{D}^n X$ is defined uniquely up to isomorphism. Note that in this case $\mathrm{D}^n X = \mathrm{D}^{n-i}\Omega^i X$ for $0 \le i \le n$, where $\Omega^i X$ denotes the ith syzygy module in a minimal free resolution of X.

Now let Y be a finitely generated module over the ring A, and

$$G : \cdots \to G_2 \to G_1 \to G_0 \to 0$$

a projective resolution of Y over A. Reindexing the complex

$$\mathrm{D}^n F : 0 \to F_0^* \xrightarrow{d_1^*} F_1^* \xrightarrow{d_2^*} \cdots \xrightarrow{d_{n+1}^*} F_{n+1}^*$$

by $(\mathrm{D}^n F)_i = F_{n-i+1}^*$ we form the first quadrant bicomplex $(\mathrm{D}^n F) \otimes_A G$. The E^2 terms corresponding to the filtration of $(\mathrm{D}^n F) \otimes_A G$ by columns are $E_{p,q}^2 = 0$ if $q > 0$, and $E_{p,0}^2 = \mathrm{Ext}_A^{n-p+1}(X, Y)$ for $p > 0$. It follows that

$$\mathrm{H}_i(\mathrm{Tot}((\mathrm{D}^n F) \otimes_A G)) \cong \mathrm{Ext}_R^{n-i+1}(X, Y)$$

for $i > 0$.

By filtering $(\mathrm{D}^n F) \otimes_A G$ along rows we get $E_{p,q}^2 = \mathrm{Tor}_p^A(\mathrm{Ext}_A^{n-q+1}(X, A), Y)$ if $q > 0$, and $E_{p,0}^2 = \mathrm{Tor}_p^A(\mathrm{D}^n X, Y)$. We now summarize the relevant information.

2.1. For all $n \ge 0$ there exists a first quadrant spectral sequence

$$E_{p,q}^2 \underset{p}{\Rightarrow} \mathrm{Ext}_A^{n-p-q+1}(X, Y),$$

where

$$E_{p,q}^2 = \begin{cases} \mathrm{Tor}_p^A(\mathrm{D}^n X, Y) & \text{if } q = 0 \\ \mathrm{Tor}_p^A(\mathrm{Ext}_A^{n-q+1}(X, A), Y) & \text{if } q > 0. \end{cases}$$

2.1.1. In particular, we have for all $n \ge 0$ the 5-term exact sequences

$$\mathrm{Ext}_A^{n-1}(X, Y) \to \mathrm{Tor}_2^A(\mathrm{D}^n X, Y) \to \mathrm{Ext}_A^n(X, A) \otimes_A Y \to \mathrm{Ext}_A^n(X, Y) \to \mathrm{Tor}_1^A(\mathrm{D}^n X, Y) \to 0.$$

2.1.2. If $E_{p,q}^2 = 0$ except for rows $q = 0$ and $q = t$ with $0 < t \le n + 1$, we have long exact sequences

$$\cdots \to \mathrm{Ext}_A^{n-i+1}(X, Y) \to \mathrm{Tor}_i^A(\mathrm{D}^n X, Y) \to \mathrm{Tor}_{i-t-1}^A(\mathrm{Ext}_A^{n-t+1}(X, A), Y) \to \mathrm{Ext}_A^{n-i+2}(X, Y) \to \cdots$$

When $t = n + 1$ and $i = n + 2$, the homomorphism above,

$$\mathrm{Hom}_A(X, A) \otimes_A Y \to \mathrm{Hom}_A(X, Y),$$

is the natural homomorphism given by $f \otimes y \mapsto \{x \mapsto f(y)x\}$.

2.1.3. If $E_{p,q}^2 = 0$ except for row $q = 0$, then we have isomorphisms

$$\mathrm{Ext}_A^i(X, Y) \cong \mathrm{Tor}_{n-i+1}^A(\mathrm{D}^n X, Y)$$

for $0 \le i \le n$.

The only statement that needs some justification is 2.1.2. The existence of the long exact sequence is a standard exercise, see, for example [18, Exercise 4, p. 332]. The second statement follows by checking the natural maps in the long exact sequence (cf. *loc. cit.*) together with the natural isomorphisms comparing the homology of $\mathrm{Hom}_A(F, A) \otimes_A Y$ with that of $\mathrm{Hom}_A(F, Y)$.

3. On the Auslander-Reiten Conjecture

In this section we let R be a commutative noetherian ring. The *Auslander-Reiten Conjecture* (in the commutative case) states that for every finitely generated R-module M,

$$\mathrm{Ext}^i_R(M, M\oplus R)=0 \text{ for all } i>0 \text{ if and only if } M \text{ is projective.}$$

In some cases where the Auslander-Reiten Conjecture is known to hold, for example, when the ring is a complete intersection [4, 1.9], or more generally, when the modules have finite complete intersection dimension [3, 4.2], the modules M satisfy the following property: there exists a positive integer n_M such that for all finitely generated R-modules N, $\mathrm{Tor}^R_j(M,N)=0$ for $i\le j\le i+n_M$ implies that $\mathrm{Tor}^R_j(M,N)=0$ for all $j\ge i$. In other words, if n_M+1 consecutive $\mathrm{Tor}^R(M,N)$ vanish, then all subsequent $\mathrm{Tor}^R(M,N)$ do as well. We show that this weaker rigidity condition is enough for the Auslander-Reiten Conjecture to hold. (See [10, 3.6] for examples of rigidity for modules of non-finite complete intersection dimension.)

The proof of the following is immediate from the exact sequences 2.1.2 for $t=n+1$.

Proposition 3.1. *Let R be a commutative noetherian ring, and M a finitely generated R-module. Fix $n\ge 1$, and suppose that $\mathrm{Ext}^i_R(M, M\oplus R)=0$ for all $1\le i\le n$. Then we have*

1. $\mathrm{Tor}^R_i(\mathrm{D}^nM, M)=0$ *for* $1\le i\le n$,
2. *and an exact sequence*

$$0\to \mathrm{Tor}^R_{n+2}(D^nM,M)\to \mathrm{Hom}_R(M,R)\otimes_R M\to \mathrm{Hom}_R(M,M)\to \mathrm{Tor}^R_{n+1}(D^nM,M)\to 0,$$

where the middle homomorphism is the natural one.

The following is like the results of Section One, but we must assume some additional $\mathrm{Ext}_R(M,R)$ vanishing.

Corollary 3.2. *Suppose that R is a commutative noetherian ring and M a finitely generated R-module for which there exists an integer n_M such that for all finitely generated R-modules N, $\mathrm{Tor}^R_i(M,N)=0$ for $j\le i\le j+n_M$ implies $\mathrm{Tor}^R_i(M,N)=0$ for all $i\ge j$. Then $\mathrm{Ext}^i_R(M,M\oplus R)=0$ for $1\le i\le n_M+1$ if and only if M is projective. In particular, the Auslander-Reiten Conjecture holds for M.*

Proof. Suppose that $\mathrm{Ext}^i(M,M\oplus R)=0$ for all $1\le i\le n_M+1$. Then from Proposition 3.1, we have $\mathrm{Tor}^R_i(\mathrm{D}^{n_M+1}M,M)=0$ for $1\le i\le n_M+1$. By assumption then $\mathrm{Tor}^R_{n_M+2}(\mathrm{D}^{n_M+1}M,M)=0$. The exact sequence in 3.1(2) now shows that the natural map $\mathrm{Hom}_R(M,R)\otimes_R M\to \mathrm{Hom}_R(M,M)$ is onto, and thus M is projective. □

For some rings R the condition $\mathrm{Ext}^i_R(M,R)=0$ for $i>0$ alone implies that M is projective:

Proposition 3.3. *Let $(R,\mathfrak{m},k)$ be a local ring for which there exists an integer d such that for every finitely generated R-module M of infinite projective dimension, one of the sequences $\{\dim_k \mathrm{Tor}^R_{2i}(M,k)\}_{2i>d}$ or $\{\dim_k \mathrm{Tor}^R_{2i+1}(M,k)\}_{2i+1>d}$ is strictly increasing. Then for every finitely generated R-module M we have*

$\operatorname{Ext}^i_R(M,R) = 0$ for $1 \le i \le 2d+5$ implies $\operatorname{pd}_R M < \infty$; if moreover $d \ge (\dim R - 5)/2$, then M is free.

If for every finitely generated R-module M the sequence $\{\dim_k \operatorname{Tor}^R_i(M,k)\}_{i>d}$ is strictly increasing, then the same conclusions hold assuming $\operatorname{Ext}^i_R(M,R) = 0$ for $1 \le i \le 2d+2$; and $d \ge (\dim R - 2)/2$

Proof. From the exact sequence 2.1.2 with $X = M$, $Y = k$, $n = 2d+5$ and $t = 2d+6$ we have for $1 \le i \le 2d+5$ the isomorphisms

$$\operatorname{Ext}^{2d-i+6}_R(M,k) \cong \operatorname{Tor}^R_i(D^{2d+5}M,k).$$

Since $\operatorname{Ext}^{2d-i+6}_R(M,k)$ and $\operatorname{Tor}^R_{2d-i+6}(M,k)$ are the same k-vector spaces for all i, we have

$$\dim_k \operatorname{Tor}^R_{2d-i+6}(M,k) = \dim_k \operatorname{Tor}^R_i(D^{2d+5}M,k) \tag{3.3.1}$$

for $1 \le i \le 2d+5$. Suppose that M has infinite projective dimension, then so does $D^{2d+5}M$. Thus, on the one hand either

$$\dim_k \operatorname{Tor}^R_{d+1}(M,k) < \dim_k \operatorname{Tor}^R_{d+3}(M,k),$$

or

$$\dim_k \operatorname{Tor}^R_{d+2}(M,k) < \dim_k \operatorname{Tor}^R_{d+4}(M,k),$$

and on the other hand either

$$\dim_k \operatorname{Tor}^R_{d+3}(D^{2d+5}M,k) < \dim_k \operatorname{Tor}^R_{d+5}(D^{2d+5}M,k),$$

or

$$\dim_k \operatorname{Tor}^R_{d+2}(D^{2d+5}M,k) < \dim_k \operatorname{Tor}^R_{d+4}(D^{2d+5}M,k).$$

In view of 3.3.1 this is a contradiction. Therefore M has finite projective dimension over R. In this case it is well-known that $\operatorname{Ext}^{\operatorname{pd}_R M}_R(M,R) \ne 0$, and so by the assumption on the vanishing of $\operatorname{Ext}^i_R(M,R)$ for $1 \le i \le 2d+5$, and if $d \ge (\dim R - 5)/2$, then we have $\operatorname{pd}_R M = 0$, that is, M is free.

The proof of the last statement is now clear. □

We give some effective results for specific rings, which add perspective to known results for the cases (1) (cf. [15, 1.4]), and (2) (cf. [12, 4.1]), below.

Corollary 3.4. *Let $(R,\mathfrak{m},k)$ be a local ring for which one of the following holds:*

1. *R is Golod, and $d = 2\mu(\mathfrak{m})$;*
2. *$\mathfrak{m}^3 = 0$ with $\dim_k \mathfrak{m}^2 > \dim_k \mathfrak{m}/\mathfrak{m}^2$, and $d \ge 1$.*
3. *$R = Q/\mathfrak{n}J$, where $(Q,\mathfrak{n})$ is a local ring, J is a non-zero ideal of Q, and $d \ge (\dim R - 5)/2$.*

Then for every finitely generated R-module M we have $\operatorname{Ext}^i_R(M,R) = 0$ for all $1 \le i \le 2d+2$ (with $2d+2$ replaced by $2d+5$ in case (3)) implies that R is free. In particular, the Auslander-Reiten conjecture holds for all finitely generated R-modules.

Proof. For case (1), Peeva shows in [19, Proposition 5] that for $i \ge 2\mu(\mathfrak{m})$ the sequence $\{\dim_k \operatorname{Tor}^R_i(M,k)\}$ is strictly increasing.

Lescot shows in [17, Theorem B] that the sequence $\{\dim_k \operatorname{Tor}^R_i(M,k)\}_{i\ge1}$ is strictly increasing in case (2). He also shows in Theorem A.2 of *loc. cit.* that the sequences $\{\dim_k \operatorname{Tor}^R_{2i}(M,k)\}_{i\ge1}$ and $\{\dim_k \operatorname{Tor}^R_{2i+1}(M,k)\}_{i\ge0}$ are strictly increasing in case (3). □

Finally we give a proof alternate to [12, 4.1] showing that the Auslander-Reiten Conjecture holds for finitely generated modules over all local rings $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$.

Proposition 3.5. *Let $(R, \mathfrak{m}, k)$ be a local ring such that $\mathfrak{m}^3 = 0$, and M a finitely generated R-module. Then $\operatorname{Ext}_i^R(M, R) = 0$ for all $i > 0$ and $\operatorname{Ext}_R^i(M, M) = 0$ for two consecutive values of $i > 0$ imply that M is free.*

Proof. Set $e = \dim_k \mathfrak{m} / \mathfrak{m}^2$ and $r = \dim_k \mathfrak{m}^2$. We assume that M is not free and derive a contradiction. The condition $\operatorname{Ext}_R^i(M, R) = 0$ for all $i > 0$ implies then four things. First that the projective dimension of M is infinite; second that $e = r + 1$ [7, Theorem A]; third that for $i \gg 0$ all $\dim_k \operatorname{Tor}_i^R(M, k)$ are equal, say, to a constant a [7, Theorem B]; and fourth that for all syzygies $\Omega^j M$ of M, we have $\operatorname{Ext}_R^i(\Omega^j M, \Omega^j M) \cong \operatorname{Ext}_R^i(M, M)$ for all $i > 0$. The fourth fact allows us to replace M by an appropriate syzygy, so that we may assume $\operatorname{Ext}_R^i(M, M) = 0$ for two consecutive values of $i > 0$, $\dim_k \operatorname{Tor}_i^R(M, k) = a$ for all $i \geq 0$, and $\mathfrak{m}^2 M = 0$. If we can show that consequently M has finite projective dimension, we will have our desired contradiction, by the first fact above.

Now suppose $\operatorname{Ext}_R^n(M, M) = \operatorname{Ext}_R^{n-1}(M, M) = 0$ for some $n > 1$. By 2.1.1 we see that $\operatorname{Tor}_1^R(D^n M, M) = \operatorname{Tor}_2^R(D^n M, M) = 0$. It follows that the Poincaré series of $D^n M \otimes_R M$ has the form $a^2 + 2a^2 t + 3a^2 t^2 + \cdots$. On the other hand, by [17, 3.3] and the second fact above, the Poincaré series of $D^n M \otimes_R M$ has the form $a + (ea - s)t + ((e^2 - e + 1)a - es)t^2 + \cdots$, where $s = \dim_k \mathfrak{m} M$. Comparing coefficients yields the equation $a = \frac{e-1}{2e-3}$, for which $e = 2$ and $a = 1$ are the only integer solutions with $a > 0$. Now $e = 2$ means that R is a complete intersection, and thus M has finite projective dimension by [3, 4.2]. ☐

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