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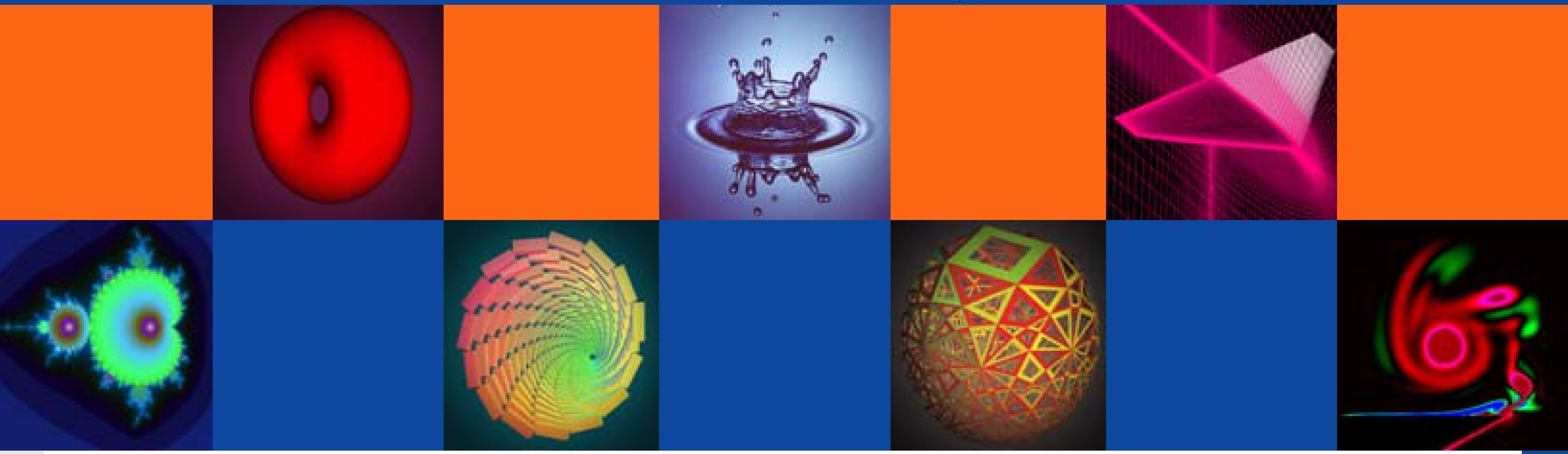

On the Blow-up Phenomena For The Degasperis-Procesi Equation

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On the blow-up phenomena for the Degasperis-Procesi equation

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Abstract

This paper is concerned with the formation of singularities for the Degasperis-Procesi equation on the line. It is shown that the lifespan of solutions to the Degaperis-Procesi equation is not affected by the smoothness or size of the initial profiles, but affected by the shape of the initial profiles. Criteria guaranteeing wave-breaking for solutions with certain smooth initial profiles are described in detail and two results of blow-up solutions are established. The exact blow-up rate and the blow-up set for a class of initial profiles are also determined.

Mathematics Subject Classification (2000): 35G25, 35L05

1 Introduction

The Degasperis-Procesi (DP) equation

$$y_t + y_x u + 3yu_x = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

with $y = u - u_{xx}$, was originally derived by Degasperis-Procesi [19] using the method of asymptotic integrability up to third order as one of three equations in the family of third order dispersive PDE conservation laws of the form

$$u_t - \alpha^2 u_{xxt} + \gamma u_{xxx} + c_0 u_x = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x. \tag{1.1}$$

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The other two integrable equations in the family are the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = 0$$

and the Camassa-Holm (CH) shallow water equation [3, 21, 27],

$$y_t + y_x u + 2yu_x = 0, \quad y = u - u_{xx}$$

These three cases exhaust in the completely integrable candidates for (1.1) by Painlevé analysis. Both the KdV equation [20] and the Camassa-Holm equation [1, 6, 8, 13, 14] are completely integrable models for the propagation of shallow water waves. The DP equation is also in dimensionless space-time variables (x, t) an approximation to the incompressible Euler equations for shallow water under the Kodama transformation [25, 26] and its asymptotic accuracy is the same as that of the Camassa-Holm shallow water equation, where $u(t, x)$ is considered as the fluid velocity at time t in the spatial x-direction with momentum density y. Degasperis, Holm and Hone [18] showed the formal integrability of the DP equation as Hamiltonian systems by constructing a Lax pair and a bi-Hamiltonian structure. The DP equation is observed a model supporting shock waves [30].

It is well known that the KdV equation is an integrable Hamiltonian equation that possesses smooth solitons as traveling waves. In the KdV equation, the leading order asymptotic balance that confines the traveling wave solitons occurs between nonlinear steepening and linear dispersion. However, the nonlinear dispersion and nonlocal balance in the CH equation and the DP equation, even in the absence of linear dispersion, can still produce a confined solitary traveling waves

$$u(t, x) = ce^{-|x-ct|},$$

traveling at constant speed $c > 0$, which are called the peakons [3, 18]. Peakons of both equations are true solitons that interact via elastic collisions under the CH dynamics, or the DP dynamics, respectively. The peakons of the CH equation are orbitally stable [17].

Note that we can rewrite the DP equation as

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0, \ x \in \mathbb{R}. \tag{1.2}$$

The peakon solitons are not classical solutions of (1.2). They satisfy the Degasperis-Procesi equation in the conservation law form

$$u_t + \partial_x \left(\frac{1}{2}u^2 + (1 - \partial_x^2)^{-1} \left(\frac{3}{2}u^2 \right) \right) = 0, \quad t > 0, \quad x \in \mathbb{R}. \tag{1.3}$$

Recently, Lundmark and Szmigielski [31] presented an inverse scattering approach for computing n-peakon solutions to Eq.(1.2). Holm and Staley [25] studied stability of solitons and peakons numerically to Eq.(1.2). Analogous to the case of Camassa-Holm equation [9], Henry [24] showed that smooth solutions to Eq.(1.2) have infinite speed of propagation.

Since its discovery, there has been considerable interest in the Deasperis-Procesi equation, cf. [14, 24, 28, 30, 32, 36, 37] and the citations therein. We shall here mention a few typical results. For example, Yin proved local well-posedness to Eq.(1.2) with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Eq.(1.2) possesses an infinite number of conservation laws, but neither of them control of the H^s-norm for $s \geq 1$. Hence these local existence results cannot be turned into global ones. The global existence of strong solutions and global weak solutions and blow-up structure to Eq.(1.2) were investigated in [38, 39]. Coclite and Karlsen [4] obtained global existence results for entropy weak solutions of Eq.(1.3) belonging to the class of $L^1(\mathbb{R}) \cap BV(\mathbb{R})$ and the class of $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$. Escher, Liu and Yin [22] also established global weak solutions in $H^1(\mathbb{R})$ and blow-up structure for Eq.(1.3).

More recently, Liu and Yin [29] proved that the first blow-up to Eq.(1.2) must occur as wave breaking and shock waves possibly appear afterwards. It is shown in [29] that the lifespan of solutions of the DP equation (1.2) is not affected by the smoothness and size of the initial profiles, but affected by the shape of the initial profiles. This can be viewed as a significant difference between the DP equation (or the CH equation) and the KdV. It is also noted that the KdV equation, unlike the CH equation or DP equation, does not have wave breaking phenomena, that is, the wave profile remains bounded, but its slope becomes unbounded in finite time [34]. For the CH equation, a procedure to understand the continuation of solutions past wave breaking has been recently presented by Bressan and Constantin in [2].

Although the DP equation is similar to the CH equation in several aspects, we would like to point out that these two equations are truly different. One of the novel features of Eq.(1.2) is it has not only peakon solitons [18], $u(t,x) = ce^{-|x-ct|}$, $c > 0$ but also shock peakons [5, 30] of the form

$$u(t,x) = -\frac{1}{t+k}\mathrm{sgn}(x)e^{-|x|},\ k > 0.$$

It is easy to see from [30] that the above shock-peakon solutions can be observed by substituting $(x,t) \longmapsto (\epsilon x, \epsilon t)$ to Eq.(1.2) and letting $\epsilon \to 0$ so that it yields the "derivative Burgers equation" $(u_t + uu_x)_{xx} = 0$, from which shock waves form. The periodic shock waves were established by Escher, Liu and Yin [23].

On the other hand, the isospectral problem in the Lax pair for Eq.(1.2) is the third-order equation

$$\psi_x - \psi_{xxx} - \lambda y \psi = 0$$

cf. [18], while the isospectral problem for the Camassa-Holm equation is the second order equation

$$\psi_{xx} - \frac{1}{4}\psi - \lambda y\psi = 0$$

(in both cases $y = u - u_{xx}$) cf. [3]. Another indication of the fact that there is no simple transformation of Eq.(1.2) into the Camassa-Holm equation is the entirely different form of conservation laws for these two equations [3, 18]. Furthermore, the Camassa-Holm equation is a re-expression of geodesic flow on the diffeomorphism group [14] or on the Bott-Virasoro group [33], while no such geometric derivation of the Degasperis-Procesi equation is available.

The following are three useful conservation laws of the Degasperis-Procesi equation.

$$E_1(u) = \int_{\mathbb{R}} y\,dx, \quad E_2(u) = \int_{\mathbb{R}} yv\,dx, \quad E_3(u) = \int_{\mathbb{R}} u^3\,dx,$$

where $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$, while the corresponding three useful conservation laws of the Camassa-Holm equation are the following:

$$F_1(u) = \int_{\mathbb{R}} y\,dx, \quad F_2(u) = \int_{\mathbb{R}} (u^2 + u_x^2)\,dx, \quad F_3(u) = \int_{\mathbb{R}} (u^3 + uu_x^2)\,dx.$$

It is found that the corresponding conservation laws of the Degasperis-Procesi equation are much weaker than those of the Camassa-Holm equation. Therefore, the issue of if and how particular initial data generate a blow-up in finite time is more subtle.

As far as we know, the case of the Camassa-Holm equation is well understood by now [7, 10, 11, 12, 15, 35] and the citations therein, while the Degasperis-Procesi equation case is the subject of this paper. The goal of this paper is to establish two new blow-up results for Eq.(1.2) and to give precise description of the blow-up set and the blow-up rate as well so that important physical phenomena of Eq.(1.2) (such as, wave breaking and shock waves) could be understood deeply. It will be seen in Section 3 that these two new blow-up results (Theorems 3.1, Theorem 3.2) based on the steepening lemma [3, 7] depend on some shape of initial profiles which are different from those in Theorem 4.2 [29].

It was assumed in [29] that there exists only one point $x_0 \in \mathbb{R}$ such that initial momentum density $y_0(x_0) = 0$. Under this assumption, it was shown that if $(x - x_0)y_0 \le 0$, then the corresponding solution to Eq.(1.2) blows up in finite time (Lemma 2.6) and if $(x - x_0)y_0 \ge 0$, then the solution exists globally (Lemma 2.7). In this paper, we prove that if the initial momentum density y_0 is odd and there exists another zero $x_0 \in [0, \infty)$ besides $x = 0$ by the oddness of y_0 such that $y_0(x_0) = 0$, then the corresponding solution to Eq.(1.2) always blows up in finite time (Theorem 3.1, Theorem 3.2). Therefore, the blow-up results established in the paper give precise descriptions of wave-breaking phenomena of the DP flow in a different direction. It will be seen in Section 3 that we use quite different methods to prove the blow-up results Theorem 3.1 and Theorem 3.2.

The remainder of the paper is organized as follows. In Section 2, we recall the local well-posedness of the Cauchy problem of Eq.(1.2) with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, the precise blow-up scenario of strong solutions, and several useful results which are crucial in the proof of blow-up phenomena for Eq.(1.2) from [36, 39]. Section 3 is devoted to establish two new blow-up results. In the last section, we give precise descriptions of the blow-up mechanism with certain initial profiles.

Notation. As above and henceforth, we denote by $*$ the convolution. For $1 \leq p \leq \infty$, the norm in the Lebesgue space $L^p(\mathbb{R})$ will be written $\|\cdot\|_{L^p}$, while $\|\cdot\|_{H^s}$, $s \geq 0$ will stand for the norm in the classical Sobolev spaces $H^s(\mathbb{R})$.

2 Preliminaries

Since we shall also use some properties of solutions in $H^s(\mathbb{R})$, $s > \frac{3}{2}$, we briefly collect the needed results from [29, 36, 39] in order to pursue our goal.

With $y := u - u_{xx}$, Eq.(1.2) takes the form of a quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} y_t + uy_x + 3u_x y = 0, & t > 0,\ x \in \mathbb{R}, \\ y(0,x) = u_0(x) - u_{0,xx}(x), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Note that if $p(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $(1-\partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathbb{R})$ and $p*(u-u_{xx}) = u$. Using this identity, we can rewrite Eq.(2.1) as follows:

$$\begin{cases} u_t + uu_x + \partial_x p * (\frac{3}{2}u^2) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.2}$$

The local well-posedness of the Cauchy problem of Eq.(2.2) with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ can be obtained by applying the Kato's theorem [36]. As a result, we have the following well-posedness result.

Lemma 2.1. *[36] Given $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, there exist a maximal $T = T(u_0) > 0$ and a unique solution u to Eq.(1.2) (or Eq.(2.2)), such that*

$$u = u(\cdot, u_0) \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : H^s(\mathbb{R}) \to C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$ is continuous and the maximal time of existence $T > 0$ can be chosen to be independent of s.

By using the local well-posedness in Lemma 2.1 and the energy method, one can get the following precise blow-up scenario of strong solutions to Eq.(2.2).

Lemma 2.2. *[36] Given $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, blow up of the solution $u = u(\cdot, u_0)$ in finite time $T < +\infty$ occurs if and only if*

$$\liminf_{t\uparrow T} \{\inf_{x\in\mathbb{R}} [u_x(t,x)]\} = -\infty.$$

Consider the following differential equation

$$\begin{cases} q_t = u(t,q), & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{2.3}$$

Applying classical results in the theory of ordinary differential equations, one can obtain the following two results on q which are crucial in the proof of global existence and blow-up solutions.

Lemma 2.3. *[39] Let $u_0 \in H^s(\mathbb{R})$, $s \geq 3$, and let $T > 0$ be the maximal existence time of the corresponding solution u to Eq.(2.2). Then the Eq.(2.3) has a unique solution $q \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(\int_0^t u_x(s, q(s,x))ds\right) > 0, \ \ \forall (t,x) \in [0,T) \times \mathbb{R}.$$

Lemma 2.4. *[39] Let $u_0 \in H^s(\mathbb{R})$, $s \geq 3$, and let $T > 0$ be the maximal existence time of the corresponding solution u to Eq.(2.2). Setting $y := u - u_{xx}$, we have*

$$y(t, q(t,x))q_x^3(t,x) = y_0(x), \quad \forall (t,x) \in [0,T) \times \mathbb{R}.$$

Let us finally present a priori estimate and a recent blow-up result for the Degasperis-Procesi equation.

Lemma 2.5. *[29] Assume $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Let T be the maximal existence time of the solution u to Eq.(2.2) guaranteed by Lemma 2.1. Then we have*

$$\|u(t,x)\|_{L^\infty} \leq 3\|u_0(x)\|_{L^2}^2 t + \|u_0(x)\|_{L^\infty}, \quad \forall t \in [0,T].$$

Lemma 2.6. *[29] Let $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$. Assume there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} y_0(x) = u_0(x) - u_{0,xx}(x) \geq 0 & \text{if} \quad x \leq x_0, \\ y_0(x) = u_0(x) - u_{0,xx}(x) \leq 0 & \text{if} \quad x \geq x_0, \end{cases}$$

and y_0 changes sign. Then, the corresponding solution to Eq.(2.2) blows up in finite time.

Lemma 2.7. *[29] Assume $u_0 \in H^s(\mathbb{R}),\ s > \frac{3}{2}$ and there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} y_0(x) \leq 0 & \text{if} \quad x \leq x_0, \\ y_0(x) \geq 0 & \text{if} \quad x \geq x_0. \end{cases}$$

Then Eq.(2.2) has a unique global strong solution

$$u = u(., u_0) \in C([0, \infty); H^s(\mathbb{R})) \cap C^1([0, \infty); H^{s-1}(\mathbb{R})).$$

Moreover, $E_2(u) = \int_{\mathbb{R}} yv\, dx$ is a conservation law, where $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$, and for all $t \in \mathbb{R}_+$ we have
(i) $u_x(t, \cdot) \geq -|u(t, \cdot)|$ on $\mathbb{R}$,
(ii) $\|u\|_1^2 \ \leq 6\|u_0\|_{L^2}^4 t^2 + 4\|u_0\|_{L^2}^2 \|u_0\|_{L^\infty} t + \|u_0\|_1^2$.

3 Blow-up results

Our purpose here is to establish two new blow-up results to Eq.(2.2) with certain initial profiles different from Lemma 2.6.

The first principal result are stated as follows.

Theorem 3.1. *Assume $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$ and $y_0(x) = u_0(x) - u_{0,xx}(x)$ is odd. If there is a $x_0 > 0$ such that*

$$\begin{cases} y_0(x) > 0 & x \in (-\infty, -x_0), \\ y_0(x) < 0 & x \in (-x_0, 0), \end{cases}$$

and $y_0(-x_0) = 0$, then the corresponding solution $u(t, x)$ to Eq.(2.2) blows up in finite time.

Proof. By Lemma 2.1 and a simple density argument, we only need to show that the above theorem holds for $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(2.2) with the initial data $u_0 \in H^3(\mathbb{R})$.

Note that

$$u(t, x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^{\eta} y(t, \eta) d\eta + \frac{e^x}{2} \int_x^{\infty} e^{-\eta} y(t, \eta) d\eta \tag{3.1}$$

and

$$u_x(t, x) = -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^{\eta} y(\eta) d\eta + \frac{e^x}{2} \int_x^{\infty} e^{-\eta} y(\eta) d\eta. \tag{3.2}$$

From the above two relations (3.1) and (3.2), we deduce that

$$\begin{aligned} u(t, x) + u_x(t, x) &= e^x \int_x^{\infty} e^{-\eta} y(t, \eta) d\eta, \\ u(t, x) - u_x(t, x) &= e^{-x} \int_{-\infty}^{x} e^{\eta} y(t, \eta) d\eta. \end{aligned} \tag{3.3}$$

Since $u_0(x) = p * y_0(x)$ and y_0 is odd, it follows that $u_0(x)$ is odd. As one can check, the function

$$v(t,x) := -u(t,-x), \quad t \in [0,T), \quad x \in \mathbb{R},$$

is also a solution of Eq.(1.2) in $C([0,T);H^3)\bigcap C^1([0,T);H^2)$ with initial data u_0. By uniqueness we conclude that $v \equiv u$. Thus, $u(t,\cdot)$ and $y(t,\cdot)$ are odd for any $t \in [0,T)$. Let $q(t,\cdot)$ be defined in (2.3). Then, $q(t,\cdot)$ is also odd for any $t \in [0,T)$.

Since the function $q(t,x)$ is an increasing diffeomorphism of $\mathbb{R}$ with $q_x(t,x) > 0$ with respect to time t, it follows from the assumption of the theorem and Lemma 2.4

$$\begin{cases} y(t,x) > 0 & x \in (-\infty, -q(t,x_0)), \\ y(t,x) < 0 & x \in (-q(t,x_0), 0), \end{cases} \tag{3.4}$$

and $y(t, q(t,-x_0)) = 0$ for all $t \in [0,T)$.

In view of (3.3) and (3.4), we have for all $t \in [0,T)$

$$(u - u_x)(t, q(t,-x_0)) = e^{-q(t,-x_0)} \int_{-\infty}^{q(t,-x_0)} e^{\eta} y(t,\eta) d\eta > 0 \tag{3.5}$$

and

$$\begin{aligned}
&(u_x + u)(t, q(t,-x_0)) = e^{q(t,-x_0)} \int_{q(t,-x_0)}^{\infty} e^{-\eta} y(t,\eta) d\eta \\
&= e^{q(t,-x_0)} \left[\left(\int_{q(t,-x_0)}^{0} + \int_{0}^{q(t,x_0)} + \int_{q(t,x_0)}^{\infty} \right) e^{-\eta} y(t,\eta) d\eta \right] \\
&= e^{q(t,-x_0)} \left(\int_{q(t,-x_0)}^{0} [e^{-\eta} - e^{\eta}] y(t,\eta) d\eta + \int_{q(t,x_0)}^{\infty} e^{-\eta} y(t,\eta) d\eta \right) \\
&\leq e^{q(t,-x_0)} \int_{q(t,x_0)}^{\infty} e^{-\eta} y(t,\eta) d\eta < 0,
\end{aligned} \tag{3.6}$$

where use has been made of the fact that $y(t,\eta) = -y(t,-\eta) < 0$ for $\eta \in (q(t,x_0), \ \infty)$.

From the above two relations (3.5) and (3.6), we may also obtain

$$u_x(t, q(t,-x_0)) < 0. \tag{3.7}$$

It then follows from (3.3) and (3.4) that for $\eta \in (-\infty, q(t,-x_0)),\ t \geq 0$

(3.8)

$$u^2(t,\eta) - u_x^2(t,\eta) = \int_{-\infty}^{\eta} e^{\xi} y(t,\xi) d\xi \int_{\eta}^{\infty} e^{-\xi} y(t,\xi) d\xi$$

$$= \int_{-\infty}^{\eta} e^{\xi} y(t,\xi) d\xi \left(\int_{\eta}^{q(t,-x_0)} e^{-\xi} y(t,\xi) d\xi + \int_{q(t,-x_0)}^{\infty} e^{-\xi} y(t,\xi) d\xi \right)$$

$$\geq \int_{-\infty}^{\eta} e^{\xi} y(t,\xi) d\xi \int_{q(t,-x_0)}^{\infty} e^{-\xi} y(t,\xi) d\xi$$

$$= \left(\int_{-\infty}^{q(t,-x_0)} e^{\xi} y(t,\xi) d\xi - \int_{\eta}^{q(t,-x_0)} e^{\xi} y(t,\xi) d\xi \right) \int_{q(t,-x_0)}^{\infty} e^{-\xi} y(t,\xi) d\xi$$

$$\geq \int_{-\infty}^{q(t,-x_0)} e^{\xi} y(t,\xi) d\xi \int_{q(t,-x_0)}^{\infty} e^{-\xi} y(t,\xi) d\xi$$

$$= u^2(t, q(t,-x_0)) - u_x^2(t, q(t,-x_0)).$$

Note that (see page 347, (5.8) in [7])

$$e^{-x} \int_{-\infty}^{x} e^{\eta} \left(u_\eta^2(t,\eta) + 2u^2(t,\eta) \right) d\eta \geq u^2(t,x)$$

and $y(t, q(t,-x_0)) = 0$ for all $t \in [0,T)$. Hence in view of (2.1), (3.5), (3.6) and (3.8), we infer from the above inequality that

(3.9)

$$\frac{d}{dt}(u - u_x)(t, q(t,-x_0))$$

$$= -q_t(t,-x_0)(u - u_x)(t, q(t,-x_0)) + e^{-q(t,x_0)} \int_{-\infty}^{q(t,-x_0)} e^{\eta} y_t(t,\eta) d\eta$$

$$= u_x^2(t, q(t,-x_0)) - \frac{3}{2} u^2(t, q(t,-x_0)) + e^{-q(t,-x_0)} \int_{-\infty}^{q(t,-x_0)} \frac{3}{2} e^{\eta} u^2(t,\eta) d\eta$$

$$= u_x^2(t, q(t,-x_0)) + \frac{1}{2} e^{-q(t,-x_0)} \int_{-\infty}^{q(t,-x_0)} e^{\eta} \left(u^2(t,\eta) - u_\eta^2(t,\eta) \right) d\eta$$

$$- \frac{3}{2} u^2(t, q(t,-x_0)) + \frac{1}{2} e^{-q(t,-x_0)} \int_{-\infty}^{q(t,-x_0)} e^{\eta} \left(u_\eta^2(t,\eta) + 2u^2(t,\eta) \right) d\eta$$

$$\geq (u_x^2 - u^2)(t, q(t,-x_0)) + \frac{1}{2} e^{-q(t,-x_0)} \int_{-\infty}^{q(t,-x_0)} e^{\eta} \left(u^2(t,\eta) - u_\eta^2(t,\eta) \right) d\eta$$

$$\geq \frac{1}{2} (u_x^2 - u^2)(t, q(t,-x_0))$$

$$= -\frac{1}{2} [(u - u_x)(u + u_x)](t, q(t,-x_0)) > 0.$$

By (3.5) and (3.9), we have

(3.10)
$$
\begin{aligned}
&\frac{d}{dt}\left(e^{q(t,-x_0)}(u-u_x)(t,q(t,-x_0))\right)\\
&= e^{q(t,-x_0)}q_t(u-u_x)(t,q(t,-x_0)) + e^{q(t,-x_0)}\frac{d}{dt}(u-u_x)(t,q(t,-x_0))\\
&\geq e^{q(t,-x_0)}(u^2-uu_x)(t,q(t,-x_0)) + \frac{1}{2}e^{q(t,-x_0)}(u_x^2-u^2)(t,q(t,-x_0))\\
&= \frac{1}{2}e^{q(t,-x_0)}(u-u_x)^2(t,q(t,-x_0))\\
&\geq \frac{1}{2}e^{q(t,-x_0)}(u-u_x)(t,q(t,-x_0))[(u-u_x)(0,-x_0)] > 0.
\end{aligned}
$$

This in turn implies that

$$
\left(e^{q(t,-x_0)}(u-u_x)(t,q(t,-x_0))\right) \geq e^{-x_0}[(u-u_x)(0,-x_0)]e^{\frac{1}{2}[(u-u_x)(0,-x_0)]t}.
$$

Thus, in view of the oddness of q and lemma 2.5, we have

$$
\begin{aligned}
&-u_x(t,q(t,-x_0))\\
&\geq [(u-u_x)(0,-x_0)]e^{\left(\frac{1}{2}[(u-u_x)(0,-x_0)]t+q(t,x_0)-x_0\right)} - u(t,q(t,-x_0)) \qquad (3.11)\\
&\geq [(u-u_x)(0,-x_0)]e^{\left(\frac{1}{2}[(u-u_x)(0,-x_0)]t-x_0\right)} - \left(3\|u_0\|_{L^2}^2 t + \|u_0\|_{L^\infty}\right),
\end{aligned}
$$

where in the last inequality of the above estimate use has been made of the L^∞ estimate of the solution u in Lemma 2.5 and $q(t,x_0) > q(t,0) = 0$. Differentiating Eq.(2.2) with respect to x, in view of $\partial_x^2 p * f = p * f - f$, we have

$$
u_{tx} + uu_{xx} = -u_x^2 + \frac{3}{2}u^2 - p * \left(\frac{3}{2}u^2\right) \leq -u_x^2 + \frac{3}{2}u^2. \qquad (3.12)
$$

Note that

$$
\begin{aligned}
\frac{du_x(t,q(t,x))}{dt} &= u_{xt}(t,q(t,x)) + u_{xx}(t,q(t,x))\frac{dq(t,x)}{dt} \qquad (3.13)\\
&= u_{tx}(t,q(t,x)) + u(t,q(t,x))u_{xx}(t,q(t,x)).
\end{aligned}
$$

By (3.12) and (3.13), we have

$$
\frac{du_x(t,q(t,-x_0))}{dt} \leq -u_x^2(t,q(t,-x_0)) + \frac{3}{2}u^2(t,q(t,-x_0)). \qquad (3.14)
$$

Suppose that the solution $u(t)$ of Eq.(2.2) exists globally in time $t \in [0,\infty)$, that is, $T=\infty$. We will show this leads to a contradiction.

Comparing (3.7) and (3.11) with a priori estimate of u in Lemma 2.5, it is easy to see that there exists $t_1 > 0$ such that

$$
u_x^2(t,q(t,-x_0) \geq 3u^2(t,q(t,-x_0)), \quad t \geq t_1. \qquad (3.15)
$$

It then follows from (3.14) and (3.15) that

$$\frac{d}{dt}f(t) \leq \frac{3}{2}u^2(t,q(t,-x_0)) - f^2(t) \leq -\frac{1}{2}f^2(t), \quad t \in [t_1,\infty). \tag{3.16}$$

where the function $f(t)$ is defined by $f(t) = u_x(t,q(t,-x_0))$. In view of (3.7), we have

$$f(t) = u_x(t,q(t,-x_0)) < 0, \quad \forall t \geq 0.$$

Thus, solving the differential inequality (3.16) yields

$$\frac{1}{f(t_1)} - \frac{1}{f(t)} + \frac{1}{2}(t-t_1) \leq 0, \quad t \geq t_0.$$

Since $-\frac{1}{f(t)} > 0$, it follows that

$$\frac{1}{f(t_1)} + \frac{1}{2}(t-t_1) < \frac{1}{f(t_1)} - \frac{1}{f(t)} + \frac{1}{2}(t-t_1) \leq 0, \quad t \geq t_0,$$

which leads to a contradiction as $t \to \infty$. This proves that $T < \infty$ and completes the proof of the theorem. □

We are now in the position to present second blow-up result.

Theorem 3.2. *Assume* $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$ *and* $y_0(x) = u_0(x) - u_{0,xx}(x)$ *is odd. If there is a* $x_0 > 0$ *such that*

$$\begin{cases} y_0(x) \leq 0 & x \in (-\infty,-x_0), \\ y_0(x) > 0 & x \in (-x_0,0), \end{cases}$$

and $y_0(-x_0) = 0$*, then the corresponding solution* $u(t)$ *to Eq.(2.2) blows up in finite time.*

The technique used here is inspired from [3, 7, 10, 11] in study of various blow-up solutions for the Camassa-Holm equation. Following their general approach to blow-up solutions to the Camassa Holm equation with some fine estimates, it enables us to establish this new blow up result for the Deasperis-Procesi equation (2.2).

Proof. As we mentioned in Theorem 3.1, we only need to show that the above theorem holds for $s = 3$. Let $T > 0$ be the maximal time of existence of the solution u to Eq.(2.2) with the initial data $u_0 \in H^3(\mathbb{R})$.

By Lemma 2.3 and Lemma 2.4, in view of the assumption of the theorem, we have

$$\begin{cases} y(t,x) \leq 0 & x \in (-\infty,-q(t,x_0)), \\ y(t,x) > 0 & x \in (-q(t,x_0),0), \end{cases} \tag{3.17}$$

and $y(t,q(t,-x_0)) = 0$ for all $t \in [0,T)$.

Next, we assume that $u(t,x)$ does not blow up in finite time, i.e. $T=\infty$. Then we will show this leads to a contradiction. Note that if there exists a $t_0>0$ such that $u_x(t,0)\le 0$, then $u(x,t)$ blows up in finite time (see Theorem 3.3 in [29, 37]). Thus, we know that $u_x(t,0)>0$ for all $t\ge 0$, that is,

$$u_x(t,0)=\int_0^\infty e^{-\xi}y(t,\xi)d\xi=-\int_{-\infty}^0 e^{\xi}y(t,\xi)d\xi>0,\quad \forall t\ge 0.$$

Since $q(t,x)=-q(t,-x)\in(-q(t,x_0),0)$, $\forall x\in[-x_0,0]$, it follows from (3.3) that $\forall x\in[-x_0,0]$,

$$\begin{aligned}&(u_x-u)(t,q(t,x))=-e^{-q(t,x)}\int_{-\infty}^{q(t,x)}e^{\eta}y(t,\eta)d\eta\\&=e^{-q(t,x)}\left(-\int_{-\infty}^0 e^{\eta}y(t,\eta)d\eta+\int_{q(t,x)}^0 e^{\eta}y(t,\eta)d\eta\right)\\&\ge e^{-q(t,x)}u_x(t,0)>0.\end{aligned}\tag{3.18}$$

and

$$\begin{aligned}&(u_x+u)(t,q(t,x))=e^{q(t,x)}\int_{q(t,x)}^{\infty}e^{-\eta}y(t,\eta)d\eta\\&=e^{q(t,x)}\left(\int_0^\infty e^{-\eta}y(t,\eta)d\eta+\int_{q(t,x)}^0 e^{-\eta}y(t,\eta)d\eta\right)\\&\ge e^{q(t,x)}u_x(t,0)>0.\end{aligned}\tag{3.19}$$

By Lemma 2.4, we have

$$y^{\frac13}(t,q(t,x))q_x(t,x)=y_0^{\frac13}(x),\quad \forall (t,x)\in\mathbb{R}_+\times\mathbb{R}.$$

In view of Hölder's inequality, we deduce that $\forall (t,\xi)\in\mathbb{R}_+\times[-x_0,0]$,

$$\begin{aligned}&\left(\int_\xi^0 (y_0(x))^{\frac13}\,dx\right)^3=\left(\int_\xi^0 (y(t,q(t,x))^{\frac13}q_x(t,x)dx\right)^3\\&=\left(\int_{q(t,\xi)}^0 (y(t,q))^{\frac13}\,dq\right)^3\\&\le\int_{q(t,\xi)}^0 y(t,q)e^q dq\left(\int_{q(t,\xi)}^0 e^{-\frac12 q}dq\right)^2\\&=((u-u_x)(t,q)e^q)\,|_{q(t,\xi)}^0\left(-2e^{-\frac12 q}\,|_{q(t,\xi)}^0\right)^2\\&=\left((u_x-u)(t,q(t,\xi))e^{q(t,\xi)}-u_x(t,0)\right)e^{-q(t,\xi)}4\left(1-e^{\frac12 q}\right)^2\\&\le 4(u_x-u)(t,q(t,\xi)).\end{aligned}\tag{3.20}$$

Thus it is deduced from (3.20) that

$$0 < c \equiv \int_{-x_0}^{0} e^{\xi} \left(\int_{\xi}^{0} (y_0(x))^{\frac{1}{3}}\, dx \right)^6 d\xi \le 4 \int_{-x_0}^{0} e^{\xi} (u_x - u)^2 (t, q(t,\xi)) d\xi, \tag{3.21}$$

where c is a constant depending only on x_0 and y_0.

By (2.1), it is easy to see that

$$y_t = -y_x u - 3 y u_x = -\left(yu + u^2 - u_x^2 \right)_x .$$

It then follows that for all $(t,x) \in [0,\infty) \times \mathbb{R}$

$$\begin{aligned} &\frac{d}{dt}\left(e^{q(t,x)}(u - u_x) \right)(t, q(t,x)) \\ &= e^{q(t,x)} q_t(t,x) y(t, q(t,x)) + \int_{-\infty}^{q(t,x)} e^{\eta} y_t(t,\eta) d\eta \\ &= e^{q(t,x)} u y(t, q(t,x)) - \int_{-\infty}^{q(t,x)} e^{\eta} \left(yu + u^2 - u_\eta^2 \right)_\eta (t,\eta) d\eta \\ &= e^{q(t,x)} (u_x^2 - u u_x - \frac{1}{2} u^2)(t, q(t,x)) + \frac{3}{2} \int_{-\infty}^{q(t,x)} e^{\eta} u^2(t,\eta) d\eta \\ &= \frac{1}{2} e^{q(t,x)} (u_x - u)^2 (t, q(t,x)) + e^{q(t,x)} \left(\left(\frac{1}{2} u_x - u^2 \right)(t, q(t,x)) \right) \\ &\quad + \frac{3}{2} \int_{-\infty}^{q(t,x)} e^{\eta} u^2(t,\eta) d\eta. \end{aligned} \tag{3.22}$$

The above relation (3.22) then yields that

$$-\frac{d}{dt} u_x(t,0) = u_x^2(t,0) + \frac{3}{2} \int_{-\infty}^{0} e^{\eta} u^2(t,\eta) d\eta. \tag{3.23}$$

This implies that

$$\int_0^{\infty} u_x^2(t,0) dt \le u_x(0,0) < +\infty, \tag{3.24}$$

where use has been made of the fact that $u_x(t,0) > 0$, for all $t \ge 0$.

Next, we claim that if $u_x(t,0) > 0$ for all $t \ge 0$, then

$$\int_0^{\infty} e^{q(t,-x_0)} (u_x - u)^2 (t, q(t,-x_0)) dt \le 2 e^{-x_0} (u_x - u)(0,-x_0) < +\infty. \tag{3.25}$$

In fact, in view of (3.2) and the oddness of y and q, we have

$$\begin{aligned}\int_{q(t,-x_0)}^{\infty} e^{-\xi}y(t,\xi)d\xi &= \int_0^{\infty} e^{-\xi}y(t,\xi)d\xi + \int_{q(t,-x_0)}^{0} e^{-\xi}y(t,\xi)d\xi \\ &= \int_0^{\infty} e^{-\xi}y(t,\xi)d\xi + \int_{-q(t,x_0)}^{0} e^{-\xi}y(t,\xi)d\xi \\ &\geq \int_0^{\infty} e^{-\xi}y(t,\xi)d\xi = u_x(t,0) > 0.\end{aligned}$$

Applying the above estimate, in view of the relation (3.3), it is inferred that for $\eta \in (-\infty,\ q(t,-x_0))$

$$\begin{aligned}&u^2(t,\eta) - u_x^2(t,\eta) = \int_{-\infty}^{\eta} e^{\xi}y(t,\xi)d\xi \int_{\eta}^{\infty} e^{-\xi}y(t,\xi)d\xi \\ &= \int_{-\infty}^{\eta} e^{\xi}y(t,\xi)d\xi \left(\int_{\eta}^{q(t,-x_0)} e^{-\xi}y(t,\xi)d\xi + \int_{q(t,-x_0)}^{\infty} e^{-\xi}y(t,\xi)d\xi\right) \\ &\geq \int_{-\infty}^{\eta} e^{\xi}y(t,\xi)d\xi \int_{q(t,-x_0)}^{\infty} e^{-\xi}y(t,\xi)d\xi \\ &= \left(\int_{-\infty}^{q(t,-x_0)} e^{\xi}y(t,\xi)d\xi - \int_{\eta}^{q(t,-x_0)} e^{\xi}y(t,\xi)d\xi\right) \int_{q(t,-x_0)}^{\infty} e^{-\xi}y(t,\xi)d\xi \\ &\geq \int_{-\infty}^{q(t,-x_0)} e^{\xi}y(t,\xi)d\xi \int_{q(t,-x_0)}^{\infty} e^{-\xi}y(t,\xi)d\xi \\ &= u^2(t,q(t,-x_0)) - u_x^2(t,q(t,-x_0)).\end{aligned}$$

Following the same proof of (3.9), one can obtain

$$\frac{d}{dt}(u-u_x)(t,q(t,-x_0)) \geq \frac{1}{2}(u_x^2-u^2)(t,q(t,-x_0)).$$

From the above inequality, it follows that

$$\begin{aligned}&\frac{d}{dt}\left(e^{q(t,-x_0)}(u_x-u)(t,q(t,-x_0))\right) \\ &= e^{q(t,-x_0)}u(u_x-u)(t,q(t,-x_0)) - e^{q(t,-x_0)}\frac{d}{dt}(u-u_x)(t,q(t,-x_0)) \\ &\leq e^{q(t,-x_0)}(uu_x-u^2)(t,q(t,-x_0)) - \frac{1}{2}e^{q(t,-x_0)}(u_x^2-u^2)(t,q(t,-x_0)) \\ &= -\frac{1}{2}e^{q(t,-x_0)}(u_x-u)^2(t,q(t,-x_0)).\end{aligned}$$

Integrating both sides of the above inequality with respect to t on $[0,T)$ for any $T>0$, in view of (3.18), we obtain

$$\frac{1}{2}\int_0^T e^{q(t,-x_0)}(u_x-u)^2(t,q(t,-x_0))dt \leq e^{-x_0}(u_x-u)(0,-x_0) < +\infty.$$

This proves (3.25). On the other hand, in view of (2.3) and (3.19), we have

$$e^q q_x = e^{q+\ln q_x} = e^{(\int_0^t (u_x+u)(t,q)dt+x)} \geq e^x, \quad \forall x \in [-x_0, 0]. \tag{3.26}$$

By (3.25) and (3.26), it then follows that

$$\begin{aligned}
&\int_0^\infty dt \int_{-x_0}^0 e^x (u_x - u)^2(t, q(t,x))dx \\
&\leq \int_0^\infty dt \int_{-x_0}^0 e^{q(t,x)}(u_x - u)^2(t, q(t,x))q_x(t,x)dx \\
&\leq \int_0^\infty dt \int_{-x_0}^0 e^{q(t,x)}[(u_x - u)^2(t,q) + 2(u_x - u)y(t,q)]q_x(t,x)dx \\
&\leq -\int_0^\infty e^q (u_x - u)^2(t,q) \mid_{q(t,-x_0)}^0 dt \\
&\leq 2e^{-x_0}(u_x - u)(0, -x_0) < \infty.
\end{aligned}$$

This contradicts (3.21). Hence it is concluded that $T < \infty$. This completes the proof of the theorem. □

Remark 3.1. *Note that initial momentum density y_0 in Theorems 3.1-3.2 can have more than one zero (three zeros), while it has only one zero in the previous blow-up result Theorem 4.2 [29]. Hence Theorem 4.2 [29] can be considered to be improved in some sense.*

4 Blow-up set and blow-up rate

Our goal in this section is to give a precise description of the blow-up mechanism with certain initial profiles. We will show that there is only one point where the slope of the solution becomes infinity exactly at breaking time. The main results in this section are in analogy with results established for the CH equation in [7, 12].

Theorem 4.1. *Assume $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$ and $y_0(x) = u_0(x) - u_{0,xx}(x)$ is odd. Let T be the maximal time of existence of the corresponding solution u to Eq.(2.2) with initial data u_0. If there is a $x_0 > 0$ such that*

$$\begin{cases} y_0(x) \leq 0 & x \in (-\infty, -x_0), \\ y_0(x) > 0 & x \in (-x_0, 0), \end{cases}$$

and $y_0(-x_0) = 0$, then the solution $u(t,x)$ blows up in finite time only at zero point. Moreover,

$$\lim_{t\to T}\left(\inf_{x\in\mathbb{R}}\{u_x(t,x)\}(T-t)\right) = \lim_{t\to T}\left(\{u_x(t,0)\}(T-t)\right) = -1$$

and

$$\lim_{t\to T}\left(\sup_{x\in\mathbb{R}}\{u_x(t,x)\}(T-t)\right)=0,$$

while the solution remains uniformly bounded.

Proof. As we mentioned before, here we only need to show that the theorem holds for $s=3$. Let $T>0$ be the maximal time of existence of the solution u to Eq.(2.2) with initial data $u_0\in H^3(\mathbb{R})$.

By Theorem 3.2, we know that corresponding solution $u(t,x)$ to Eq.(2.2) with initial data u_0 blows up in finite time, i.e. , $T<\infty$. Following the same proof of Theorem 3.1 in [22], we can easily obtain that

$$\lim_{t\to T}\left(\inf_{x\in\mathbb{R}}\{u_x(t,x)\}(T-t)\right)=-1. \tag{4.1}$$

Next, we give a precise description of the blow-up mechanism. Since y_0 is odd, as before, we see that $u(t,x)$ and $y(t,x)$ are odd and $u_x(t,x)$ is even. Then we have for $(t,x)\in[0,T)\times\mathbb{R}_+$

$$\begin{aligned} u(t,x)=p*y(t,x)&=\frac{1}{2}\int_{\mathbb{R}}e^{-|x-\eta|}y(t,\eta)d\eta\\ &=\sinh(x)\int_x^\infty e^{-\eta}y(t,\eta)d\eta+e^{-x}\int_0^x\sinh(\eta)y(t,\eta)d\eta \end{aligned}\tag{4.2}$$

and

$$\begin{aligned} u_x(t,x)&=\partial_x\left[\frac{1}{2}\int_{\mathbb{R}}e^{-|x-\eta|}y(t,\eta)d\eta\right]\\ &=\cosh(x)\int_x^\infty e^{-\eta}y(t,\eta)d\eta-e^{-x}\int_0^x\sinh(\eta)y(t,\eta)d\eta. \end{aligned}\tag{4.3}$$

In particular, in view of the assumption of the theorem, we have

$$\begin{aligned} u_x(t,q(t,x_0))=\cosh(q(t,x_0))&\int_{q(t,x_0)}^\infty e^{-\eta}y(t,\eta)d\eta\\ &-e^{-q(t,x_0)}\int_0^{q(t,x_0)}\sinh(\eta)y(t,\eta)d\eta>0. \end{aligned}\tag{4.4}$$

For $x>q(t,x_0)$, we deduce from (4.2) and (4.3) that

$$\begin{aligned} u_x(t,x)&=\cosh(x)\int_x^\infty e^{-\eta}y(t,\eta)d\eta-e^{-x}\int_0^x\sinh(\eta)y(t,\eta)d\eta\\ &=(\cosh(x)+\sinh(x))\int_x^\infty e^{-\eta}y(t,\eta)d\eta-u(t,x)\\ &\geq-\mid u(t,x)\mid\geq-(3\|u_0\|_{L^2}^2T+\|u_0\|_{L^\infty}). \end{aligned}\tag{4.5}$$

For $0 < x < q(t, x_0)$, it follows from (4.2) and (4.3) that

$$
\begin{aligned}
u_x(t,x) &= \cosh(x)\int_x^{\infty} e^{-\eta} y(t,\eta)d\eta - e^{-x}\int_0^x \sinh(\eta)y(t,\eta)d\eta \\
&= \frac{\cosh(x)}{\sinh(x)}u(t,x) - \left(\frac{\cosh(x)}{\sinh(x)}+1\right)e^{-x}\int_0^x \sinh(\eta)y(t,\eta)d\eta \\
&\geq -\frac{\cosh(x)}{\sinh(x)} \mid u(t,x) \mid \geq -\frac{\cosh(x)}{\sinh(x)}(3\|u_0\|_{L^2}^2 T + \|u_0\|_{L^\infty}).
\end{aligned} \tag{4.6}
$$

From (4.4)-(4.6), we see that for any $x \neq 0$, the slope $u_x(t,x)$ has a lower bounded on $[0, T)$.

On the other hand, in view of Lemma 2.5 and $p * \left(\frac{3}{2}u^2\right)(t, q(t,x)) \geq 0$, we have

$$
\begin{aligned}
\frac{du_x(t,q(t,x))}{dt} &= u_{xt}(t,q(t,x)) + u_{xx}(t,q(t,x))\frac{dq(t,x)}{dt} \\
&= -u_x^2(t,q(t,x)) + \frac{3}{2}u^2(t,q(t,x)) - p * \left(\frac{3}{2}u^2(t,q(t,x)\right) \\
&\leq \frac{3}{2}u^2(t,q(t,x)) \leq \frac{3}{2}\left(3\|u_0\|_{L^2}^2 t + \|u_0\|_{L^\infty}\right)^2.
\end{aligned}
$$

It then follows that for all $(t,x) \in [0,T) \times \mathbb{R}$,

$$
u_x(t,q(t,x)) \leq u_x(0,x) + \frac{3T}{2}\left(3\|u_0\|_{L^2}^2 T + \|u_0\|_{L^\infty}\right)^2. \tag{4.7}
$$

The above inequality shows that for any $x \in \mathbb{R}$, the slope $u_x(t,x)$ has a upper bound on $[0,T)$. Thus, we obtain that the wave breaks in finite time exact at zero and nowhere else.

We finally give the detail description of the blow-up rate. Referring to (4.1) and (4.4)-(4.6), in view of Lemma 2.5, we have

$$
\lim_{t\to T}\left(\{u_x(t,0)\}(T-t)\right) = \lim_{t\to T}\left(\inf_{x\in\mathbb{R}}\{u_x(t,x)\}(T-t)\right) = -1,
$$

while the solution remains uniformly bounded. In addition, by (4.7), we also have

$$
\sup_{x\in\mathbb{R}}\{u_x(t,x)\} \leq u_x(0,x) + \frac{3T}{2}\left(3\|u_0\|_{L^2}^2 T + \|u_0\|_{L^\infty}\right)^2, \quad \forall t \in [0,T].
$$

The above inequality implies

$$
\lim_{t\to T}\left(\sup_{x\in\mathbb{R}}\{u_x(t,x)\}(T-t)\right) = 0.
$$

This completes the proof of the theorem. □

Remark 4.1. *Theorem 3.2 improves the recent result of Theorem 3.3 in [22].*

By Theorem 3.1 and Lemma 2.2, in view of (4.7), one can easily have the following result.

Theorem 4.2. *Assume* $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$ *and* $y_0(x) = u_0(x) - u_{0,xx}(x)$ *is odd. Let* T *be the maximal time of existence of the corresponding solution* u *to Eq.(2.2) with initial data* u_0*. If there is a* $x_0 > 0$ *such that*

$$\begin{cases} y_0(x) > 0 & x \in (-\infty, -x_0), \\ y_0(x) < 0 & x \in (-x_0, 0), \end{cases}$$

and $y_0(-x_0) = 0$*, then* $T < \infty$*. Moreover,*

$$\lim_{t \to T} \left(\inf_{x \in \mathbb{R}} \{u_x(t,x)\}(T-t) \right) = -1 \text{ and } \lim_{t \to T} \left(\sup_{x \in \mathbb{R}} \{u_x(t,x)\}(T-t) \right) = 0,$$

while the solution remains uniformly bounded.

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