Stability of Peakons for the Degasperis-Procesi Equation

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Abstract

The Degasperis-Procesi equation can be derived as a member of a one-parameter family of asymptotic shallow water approximations to the Euler equations with the same asymptotic accuracy as that of the Camassa-Holm equation. It is noted that the Degasperis-Procesi equation, unlike the Camassa-Holm equation, has not only peakon solitons but also shock peakons. In this paper, we study the orbital stability problem of the peaked solitons to the Degasperis-Procesi equation on the line. By constructing a Liapunov function, we prove that the shapes of these peakon solitons are stable under small perturbations.

Keywords: Stabiltiy; Degasperis-Procesi equation; Peakons

Mathematics Subject Classification (2000): 35G35, 35Q51, 35G25, 35L05

1 Introduction

The Degasperis-Procesi (DP) equation

$$y_t + y_x u + 3 y u_x = 0, \quad x \in \mathbb{R}, \ \ t > 0, \tag{1.1}$$

with $y = u - u_{xx}$, was originally derived by Degasperis-Procesi [13] using the method of asymptotic integrability up to third order as one of three equations in the family of third order dispersive PDE conservation laws of the form

$$u_t - \alpha^2 u_{xxt} + \gamma u_{xxx} + c_0 u_x = (c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx})_x. \tag{1.2}$$

The other two integrable equations in the family, after rescaling and applying a Galilean transformation, are the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + u u_x = 0$$

and the Camassa-Holm (CH) shallow water equation [2, 14, 17, 21],

$$y_t + y_x u + 2 y u_x = 0, \quad y = u - u_{xx}. \tag{1.3}$$

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These three cases exhaust in the completely integrable candidates for (1.2) by Painlevé analysis. Degasperis, Holm and Hone [12] showed the formal integrability of the DP equation as Hamiltonian systems by constructing a Lax pair and a bi-Hamiltonian structure.

The Camassa-Holm equation was first derived by Fokas and Fuchassteiner [17] as a bi-Hamiltonian system, and then as a model for shallow water waves by Camassa and Holm [2]. The DP equation is also in dimensionless space-time variables (x, t) an approximation to the incompressible Euler equations for shallow water under the Kodama transformation [19, 20] and its asymptotic accuracy is the same as that of the Camassa-Holm (CH) shallow water equation, where $u(t, x)$ is considered as the fluid velocity at time t in the spatial x-direction with momentum density y.

Recently, Liu and Yin [23] proved that the first blow-up in finite time to equation (1.1) must occur as wave breaking and shock waves possibly appear afterwards. It is shown in [23] that the lifespan of solutions of the DP equation (1.1) is not affected by the smoothness and size of the initial profiles, but affected by the shape of the initial profiles. This can be viewed as a significant difference between the DP equation (or the CH equation) and the KdV. It is also noted that the KdV equation, unlike the CH equation or DP equation, does not have wave breaking phenomena, that is, wave remains bounded, but its slope becomes unbounded in finite time [28].

It is well known that the KdV equation is an integrable Hamiltonian equation that possesses smooth solitons as traveling waves. In the KdV equation, the leading order asymptotic balance that confines the traveling wave solitons occurs between nonlinear steepening and linear dispersion. However, the nonlinear dispersion and nonlocal balance in the CH equation and the DP equation, even in the absence of linear dispersion, can still produce a confined solitary traveling waves

$$u(t, x) = c\varphi(x - ct) \tag{1.4}$$

traveling at constant speed $c > 0$, where $\varphi(x) = e^{-|x|}$. Because of their shape (they are smooth except for a peak at their crest), these solutions are called the peakons [2, 12]. Peakons of both equations are true solitons that interact via elastic collisions under the CH dynamics, or the DP dynamics, respectively. The peakons of the CH equation are orbitally stable [11]. For waves that approximate the peakons in a special way, a stability result was proved by a variation method [10].

Note that we can rewrite the DP equation as

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad t > 0, \ x \in \mathbb{R}. \tag{1.5}$$

The peaked solitons are not classical solutions of (1.5). They satisfy the Degasperis-Procesi equation in the conservation law form

$$u_t + \partial_x \left(\frac{1}{2}u^2 + \frac{1}{2}\varphi * \left(\frac{3}{2}u^2 \right) \right) = 0, \quad t > 0, \ x \in \mathbb{R}, \tag{1.6}$$

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where $*$ stands for convolution with respect to the spatial variable $x \in \mathbb{R}$. This is the exact meaning in which the peakons are solutions.

Recently, Lundmark and Szmigielski [25] presented an inverse scattering approach for computing n-peakon solutions to equation (1.5). Holm and Staley [19] studied stability of solitons and peakons numerically to equation (1.5). Analogous to the case of Camassa-Holm equation [6], Henry [18] showed that smooth solutions to equation (1.5) have infinite speed of propagation.

The following are three useful conservation laws of the Degasperis-Procesi equation.

$$E_1(u) = \int_{\mathbb{R}} y \, dx, \quad E_2(u) = \int_{\mathbb{R}} yv \, dx, \quad E_3(u) = \int_{\mathbb{R}} u^3 \, dx,$$

where $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$, while the corresponding three useful conservation laws of the Camassa-Holm equation are the following:

$$F_1(u) = \int_{\mathbb{R}} y \, dx, \quad F_2(u) = \int_{\mathbb{R}} (u^2 + u_x^2) \, dx, \quad F_3(u) = \int_{\mathbb{R}} (u^3 + uu_x^2) \, dx. \quad (1.7)$$

The stability of solitary waves is one of the fundamental qualitative properties of the solutions of nonlinear wave equations. Numerical simulations [12, 24] suggest that the sizes and velocities of the peakons do not change as a result of collision so these patterns are expected to be stable. Furthermore, it is observed that the shape of the peakons remains approximately the same as time evolves. As far as we know, the case of stability of the peakons for the Camassa-Holm equation is well understood by now [10, 11], while the Degasperis-Procesi equation case is the subject of this paper. The goal of this paper is to establish a stability result of peaked solitons for equation (1.5).

It is found that the corresponding conservation laws of the Degasperis-Procesi equation are much weaker than those of the Camassa-Holm equation. In particular, one can see that the conservation law $E_2(u)$ for the DP equation is equivalent to $\|u\|_{L^2}^2$. In fact, by the Fourier transform, we have

$$E_2(u) = \int_{\mathbb{R}} yv dx = \int_{\mathbb{R}} \frac{1 + \xi^2}{4 + \xi^2} |\hat{u}(\xi)|^2 d\xi \sim \|\hat{u}\|_{L^2}^2 = \|u\|_{L^2}^2. \quad (1.8)$$

Therefore, the stability issue of the peaked solitons of the DP equation is more subtle .

For the DP equation, we can only expect to obtain the orbital stability of peakons in the sense of L^2-norm due to a weaker conservation law E_2. The solutions of the DP equation usually tend to be oscillations which spread out spatially in a quite complicated way. In general, a small perturbation of a solitary wave can yield another one with a different speed and phase shift. We define the orbit of traveling-wave solutions $c\varphi$ to be the set $U(\varphi) = \{c\varphi(\cdot + x_0), \ x_0 \in \mathbb{R}\}$, and a peaked soliton of the DP equation is called orbitally stable if a wave starting close to the peakon remains close to some translate of it at all later times.

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Let us denote
$$E_2(u) = \|u\|_X^2.$$

The following stability theorem is the principal result of the present paper.

Theorem 1 (Stability) *Let $c\varphi$ be the peaked soliton defined in (1.4) traveling with speed $c > 0$. Then $c\varphi$ is orbitally stable in the following sense. If $u_0 \in H^s$ for some $s > 3/2$, $y_0 = u_0 - \partial_x^2 u_0$ is a nonnegative Radon measure of finite total mass, and*

$$\|u_0 - c\varphi\|_X < c\varepsilon, \quad |E_3(u_0) - E_3(c\varphi)| < c^3\varepsilon, \quad 0 < \varepsilon < \frac{1}{2},$$

then the corresponding solution $u(t)$ of equation (1.5) with initial value $u(0) = u_0$ satisfies

$$\sup_{t \geq 0} \|u(t, \cdot) - c\varphi(\cdot - \xi_1(t))\|_X < 3c \; \varepsilon^{1/4},$$

where $\xi_1(t) \in \mathbb{R}$ is the maximum point of the function $v(t, \cdot) = (4 - \partial_x^2)^{-1} u(t, \cdot)$. Moreover, let

$$M_1(t) = v(t, \xi_1(t)) \geq M_2(t) \cdots \geq M_n(t) \geq 0 \text{ and } m_1(t) \geq \cdots \geq m_{n-1}(t) \geq 0$$

be all local maxima and minima of the nonnegative function $v(t, \cdot)$, respectively. Then

$$\left| M_1(t) - \frac{c}{6} \right| \leq c\sqrt{2\varepsilon} \tag{1.9}$$

and

$$\sum_{i=2}^{n} \left(M_i^2(t) - m_{i-1}^2(t) \right) < 2c^2\sqrt{\varepsilon}. \tag{1.10}$$

Remark 1 *For an initial profile $u_0 \in H^s$, $s > 3/2$, there exists a local solution $u \in C([0, T), H^s)$ of (1.5) with initial data $u(0) = u_0$ [29]. Under the assumption $y_0 = u_0 - \partial_x^2 u_0 \geq 0$ in Theorem 1, the existence is global in time [23], that is $T = +\infty$. For peakons $c\varphi$ with $c > 0$, we have $\left(1 - \partial_x^2\right)(c\varphi) = 2c\delta$ (here δ is the Dirac distribution). Hence the assumption on y_0 that it is a nonnegative measure is quite natural for a small perturbation of the peakons. Existence of global weak solution in H^1 of the DP equation is also proved in [15]. Note that peakons $c\varphi$ are not strong solutions, since $\varphi \in H^s$, only for $s < 3/2$.*

The above theorem of orbital stability states that any solution starting close to peakons $c\varphi$ remains close to some translate of $c\varphi$ in the norm $\| \; \|_X$, at any later time. More information about this stability is contained in (1.9) and (1.10). Notice that for peakons $c\varphi$, the function $v_{c\varphi}$ is single-humped with the height $\frac{1}{6}c$. So (1.9) and (1.10) imply that the graph of $v(t, \cdot)$ is close to that of the peakon $c\varphi$ with a fixed $c > 0$ for all times.

There are two standard methods to study stability issues of dispersive wave equations. One is the variational approach which constructs the solitary waves

as energy minimizers under appropriate constraints, and the stability automatically follows. However, without uniqueness of the minimizer, one can only obtain the stability of the set of minima. The variational approach is used in [10] for the CH equation. It is shown in [10] that the each peakon $c\varphi$ is the unique minimum (*ground state*) of constrained energy, from which its orbital stability is proved for initial data $u_0 \in H^3$ with $y_0 = (1 - \partial_x^2)u_0 \geq 0$. Their proof strongly relies on the fact that the conserved energy F_2 in (1.7) of the CH equation is the H^1-norm of the solution. However, for the DP equation the energy E_2 in (1.8) is only the L^2 norm of the solution. Consequently, it is more difficult to use such a variational approach for the DP equation.

Another approach to study stability is to linearize the equation around the solitary waves, and it is commonly believed that nonlinear stability is governed by the linearized equation. However, for the CH and DP equations, the nonlinearity plays the dominant role rather than being a higher-order correction to linear terms. Thus it is unclear how one can get nonlinear stability of peakons by studying the linearized problem. Morover, the peaked solitons $c\varphi$ are not differentiable, which makes it difficult to analyze the spectrum of the linearized operator around $c\varphi$.

To establish the stability result for the DP equation, we extend the approach in [11] for the CH equation. The idea in [11] is to directly use the energy F_2 as the Liapunov functional. By expanding F_2 in (1.7) around the peakon $c\varphi$, the error term is in the form of the difference of the maxima of $c\varphi$ and the perturbed solution u. To estimate this difference, they establish two integral relations

$$\int g^2 = F_2(u) - 2(\max u)^2 \quad \text{and} \quad \int ug^2 = F_3(u) - \frac{4}{3}(\max u)^3$$

with a function g. Relating these two integrals, one can get

$$F_3(u) \leq MF_2(u) - \frac{2}{3}M^3, \quad M = \max u(x)$$

and the error estimate $|M - \max \varphi|$ then follows from the structure of the above polynomial inequality.

To extend the above approach to nonlinear stability of the DP peakons, we have to overcome several difficulties. By expanding the energy $E_2(u)$ around the peakon $c\varphi$, the error term turns out to be $\max v_{c\varphi} - \max v_u$, with $v_u = (4 - \partial_x^2)^{-1}u$. We can derive the following two integral relations for $M_1 = \max v_u$, $E_2(u)$ and $E_3(u)$ by

$$\int g^2 = E_2(u) - 12M_1^2 \quad \text{and} \quad \int hg^2 = E_3(u) - 144M_1^3$$

with some functions g and h related to v_u. To get the required polynomial inequality from the above two identities, we need to show $h \leq 18 \max v_u$. However, since h is of the form $-\partial_x^2 v_u \pm 6\partial_x v_u + 16v_u$, generally it can not be bounded by v_u. This new difficulty is due to the more complicated nonlinear structure and weaker conservation laws of the DP equation. To overcome it, we

introduce a new idea. By constructing g and h piecewise according to monotonicity of the function v_u, we then establish two new integral identities (3.7) and (3.9) for E_2, E_3 and all local maxima and minima of v_u. The crucial estimate $h \leq 18 \max v_u$ can now be shown by using this monotonicity structure and properties of the DP solutions. This results in inequality (3.13) related to E_2, E_3 and all local maxima and minima of v_u. By analyzing the structure of equality (3.13), we can obtain not only the error estimate $|M_1 - \max v_{c\varphi}|$ but more precise stability information from (1.10). We note that the same approach can also be used for the CH equation to gain more stability information (see Remark 2).

Although the DP equation is similar to the CH equation in several aspects, we would like to point out that these two equations are truly different. One of the novel features of the DP equation is it has not only peaked solitons [12], $u(t,x) = ce^{-|x-ct|}$, $c > 0$ but also shock peakons [4, 24] of the form

$$u(t,x) = -\frac{1}{t+k}\text{sgn}(x)e^{-|x|}, \; k > 0. \tag{1.11}$$

It is noted that the above shock-peakon solutions [24] can be observed by substituting $(x,t) \longmapsto (\epsilon x, \epsilon t)$ to equation (1.5) and letting $\epsilon \to 0$ so that it yields the "derivative Burgers equation" $(u_t + uu_x)_{xx} = 0$, from which shock waves form. The periodic shock waves were established by Escher, Liu and Yin [16].

The shock peakons can be also observed from the collision of the peakons (moving to the right) and antipeakons (moving to left) [24].

For example, if we choose the initial data

$$u_0(x) = c_1 e^{-|x-x_1|} - c_1 e^{-|x-x_2|},$$

with $c_1 > 0$, and $x_1 + x_2 = 0, x_2 > 0$, then the collision occurs at $x = 0$ and the solution

$$u(x,t) = p_1(t)e^{-|x-q_1(t)|} + p_2(t)e^{-|x-q_2(t)|},$$

$(x,t) \in \mathbb{R}_+ \times \mathbb{R}$, only satisfies the DP equation for $t < T$. The unique continuation of $u(x,t)$ into an entropy weak solution is then given by the stationary decaying shock peakon

$$u(x,t) = \frac{-sgn(x)e^{-|x|}}{k + (t-T)} \quad \text{for } t \geq T.$$

On the other hand, the isospectral problem in the Lax pair for equation (1.5) is the third-order equation

$$\psi_x - \psi_{xxx} - \lambda y \psi = 0$$

cf. [12], while the isospectral problem for the Camassa-Holm equation is the second order equation

$$\psi_{xx} - \frac{1}{4}\psi - \lambda y \psi = 0$$

(in both cases $y = u - u_{xx}$) cf. [2]. Another indication of the fact that there is no simple transformation of equation (1.5) into the Camassa-Holm equation is

the entirely different form of conservation laws for these two equations [2, 12]. Furthermore, the Camassa-Holm equation is a re-expression of geodesic flow on the diffeomorphism group [9] or on the Bott-Virasoro group [27], while no such geometric derivation of the Degasperis-Procesi equation is available.

The remainder of the paper is organized as follows. In Section 2, we recall the local well-posedness of the Cauchy problem of equation (1.5), the precise blow-up scenario of strong solutions, and several useful results which are crucial in the proof of stability theorem for equation (1.5) from [29, 30]. Section 3 is devoted to the proof of the stability result (Theorem 1).

Notation. As above and henceforth, we denote by $*$ convolution with respect to the spatial variable $x \in \mathbb{R}$. We use $\| \cdot \|_{L^p}$ to denote the norm in the Lebesgue space $L^p(\mathbb{R})$ $(1 \le p \le \infty)$, and $\| \cdot \|_{H^s}$, $s \ge 0$ for the norm in the Sobolev spaces $H^s(\mathbb{R})$.

2 Preliminaries

In the present section, we discuss the issue of well-posedness. The local existence theory of the initial-value problem is necessary for our study of nonlinear stability. We briefly collect the needed results from [23, 29, 30].

Denote $p(x) := \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1}f = p * f$ for all $f \in L^2(\mathbb{R})$ and $p * (u - u_{xx}) = u$. Using this identity, we can rewrite the DP equation (1.5) as follows:

$$u_t + uu_x + \partial_x p * \left(\frac{3}{2}u^2 \right) = 0, \quad t > 0, \ x \in \mathbb{R}. \tag{2.1}$$

The local well-posedness of the Cauchy problem of equation (1.5) with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ can be obtained by applying Kato's theorem [22, 29]. As a result, we have the following well-posedness result.

Lemma 2.1 *[29] Given $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, there exist a maximal $T = T(u_0) > 0$ and a unique solution u to equation (1.5) (or equation (2.1)), such that*

$$u = u(\cdot, u_0) \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : H^s(\mathbb{R}) \to C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R}))$ is continuous and the maximal time of existence $T > 0$ can be chosen to be independent of s.

The following two lemmas show that the only way that a classical solution to (1.5) may fail to exist for all time is that the wave may break.

Lemma 2.2 *[29] Given $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, blow up of the solution $u = u(\cdot, u_0)$ in finite time $T < +\infty$ occurs if and only if*

$$\liminf_{t \uparrow T} \{ \inf_{x \in \mathbb{R}} [u_x(t, x)] \} = -\infty.$$

Lemma 2.3 *[23] Assume $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. Let T be the maximal existence time of the solution u to equation (1.5). Then we have*

$$\|u(t,x)\|_{L^\infty} \leq 3\|u_0(x)\|_{L^2}^2 t + \|u_0(x)\|_{L^\infty}, \quad \forall t \in [0,T].$$

Now consider the following differential equation

$$\begin{cases} q_t = u(t,q), & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{2.2}$$

Applying classical results in the theory of ordinary differential equations, one can obtain the following two results on q which are crucial in the proof of global existence and blow-up solutions.

Lemma 2.4 *[30] Let $u_0 \in H^s(\mathbb{R})$, $s \geq 3$, and let $T > 0$ be the maximal existence time of the corresponding solution u to equation(1.5). Then the equation (2.2) has a unique solution $q \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = \exp\left(\int_0^t u_x(s,q(s,x))ds\right) > 0, \ \forall(t,x) \in [0,T) \times \mathbb{R}.$$

Lemma 2.5 *[30] Let $u_0 \in H^s(\mathbb{R})$, $s \geq 3$, and let $T > 0$ be the maximal existence time of the corresponding solution u to equation (2.2). Setting $y := u - u_{xx}$, we have*

$$y(t,q(t,x))q_x^3(t,x) = y_0(x), \quad \forall(t,x) \in [0,T) \times \mathbb{R}.$$

The next two lemmas clearly show that the solution of equation (1.5) is affected by the shape of the initial profiles, not the smoothness and size of the initial profiles.

Lemma 2.6 *[23] Let $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$. Assume there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} y_0(x) = u_0(x) - u_{0,xx}(x) \geq 0 & if \quad x \leq x_0, \\ y_0(x) = u_0(x) - u_{0,xx}(x) \leq 0 & if \quad x \geq x_0, \end{cases}$$

and y_0 changes sign. Then, the corresponding solution to equation(1.5) blows up in a finite time.

Lemma 2.7 *[23] Assume $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ and there exists $x_0 \in \mathbb{R}$ such that*

$$\begin{cases} y_0(x) \leq 0 & if \quad x \leq x_0, \\ y_0(x) \geq 0 & if \quad x \geq x_0. \end{cases}$$

Then equation (1.5) has a unique global strong solution

$$u = u(.,u_0) \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R})).$$

Moreover, $E_2(u) = \int_\mathbb{R} yv\,dx$ is a conservation law, where $y = (1 - \partial_x^2)u$ and $v = (4 - \partial_x^2)^{-1}u$, and for all $t \in \mathbb{R}_+$ we have
(i) $u_x(t,\cdot) \geq -|u(t,\cdot)|$ on $\mathbb{R}$,
(ii) $\|u\|_1^2 \leq 6\|u_0\|_{L^2}^4 t^2 + 4\|u_0\|_{L^2}^2\|u_0\|_{L^\infty} t + \|u_0\|_1^2.$

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The following lemma is a special case of Lemma 2.7.

Lemma 2.8 *[23] Assume $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. If $y_0 = u_0 - u_{0,xx} \geq 0 \ (\leq 0)$ on $\mathbb{R}$, then equation (1.5) has a unique global strong solution u such that $u(t,x) \geq 0 \ (\leq 0)$ and $y(t,x) = u - \partial_x^2 u \geq 0 \ (\leq 0)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.*

Lemma 2.9 *Assume $u_0 \in H^s$, $s > 3/2$ and $y_0 \geq 0$. If $k_1 \geq 1$, then the corresponding solution u of (1.5) with the initial data $u(0) = u_0$ satisfies*

$$(k_1 \pm \partial_x)u(t,x) \geq 0, \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Proof. By Lemma 2.4 and a simple density argument, it suffices to show the lemma for $s = 3$. In view of Lemma 2.5, the potential $y(t,x) = (1 - \partial_x^2)u \geq 0$, $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Note $u = (1 - \partial_x^2)^{-1}y$. Then we have

$$u(t,x) = \frac{e^{-x}}{2} \int_{-\infty}^x e^\eta y(t,\eta)d\eta + \frac{e^x}{2} \int_x^\infty e^{-\eta} y(t,\eta)d\eta \tag{2.3}$$

and

$$u_x(t,x) = -\frac{e^{-x}}{2} \int_{-\infty}^x e^\eta y(t,\eta)d\eta + \frac{e^x}{2} \int_x^\infty e^{-\eta} y(t,\eta)d\eta. \tag{2.4}$$

It then follows from the above two relations (2.3) and (2.4) that

$$(k_1 \pm \partial_x)u = \frac{1}{2}(k_1 \mp 1)e^{-x} \int_{-\infty}^x e^\eta y d\eta + \frac{1}{2}(k_1 \pm 1)e^x \int_x^\infty e^{-\eta} y d\eta \geq 0. \tag{2.5}$$

$\square$

Lemma 2.10 *Let $w(t,x) = (k_1 \pm \partial_x)u(t,x)$. Assume $u_0 \in H^s$, $s > 3/2$ and $y_0 \geq 0$. If $k_1 \geq 1$ and $k_2 \geq 2$, then we have*

$$(k_2 \pm \partial_x)(4 - \partial_x^2)^{-1}w \geq 0.$$

Proof. In view of Lemma 2.9, we have $w(t,x) \geq 0, \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}$. A simple calculation shows

$$(4 - \partial_x^2)^{-1}w = \frac{1}{4} \int_{-\infty}^\infty e^{-2|x-\xi|} w(t,\xi)d\xi$$

$$= \frac{1}{4} e^{-2x} \int_{-\infty}^x e^{2\xi} w(t,\xi)d\xi + \frac{1}{4} e^{2x} \int_x^\infty e^{-2\xi} w(t,\xi)d\xi$$

and

$$\partial_x\left((4 - \partial_x^2)^{-1}w\right) = -\frac{1}{2} e^{-2x} \int_{-\infty}^x e^{2\xi} w(t,\xi)d\xi + \frac{1}{2} e^{2x} \int_x^\infty e^{-2\xi} w(t,\xi)d\xi.$$

Combining above two identities, we get

$$(k_2 \pm \partial_x)(4 - \partial_x^2)^{-1}w = \frac{1}{4} (k_2 \mp 2)e^{-2x} \int_{-\infty}^x e^{2\xi} w(t,\xi)d\xi$$

$$+ \frac{1}{4} (k_2 \pm 2)e^{2x} \int_x^\infty e^{-2\xi} w(t,\xi)d\xi \geq 0.$$

$\square$

3 Proof of stability

In this primary section of the paper, we prove the stability theorem (Theorem 1) stated in the introduction. Note that the assumptions on the initial profiles guarantee the existence of an unique global solution of equation (1.5). The stability theorem provides a quantitative estimate of how closely the wave must approximate the peakon initially in order to be close enough to some translate of the peakon at any later time. That translate must be located at a point where the wave is tallest. The proof of Theorem 1 is based on a series of lemmas including some in the previous section.

We take the wave speed $c = 1$ and the case of general c follows by scaling the estimates.

Note that $\varphi(x) = e^{-|x|} \in H^1(\mathbb{R})$ has the peak at $x = 0$ and

$$E_3(\varphi) = \int_{-\infty}^{\infty} e^{-3|x|} dx = \frac{2}{3}. \tag{3.1}$$

Define $v_u = (4 - \partial_x^2)^{-1} u = \frac{1}{4} e^{-2|x|} * u$. Then

$$v_\varphi(x) = \frac{1}{4} \int_{\mathbb{R}} e^{-2|\eta - x|} e^{-|\eta|} d\eta = \frac{1}{3} e^{-|x|} - \frac{1}{6} e^{-2|x|}, \quad \forall x \in \mathbb{R}, \tag{3.2}$$

and thus

$$\max_{x \in \mathbb{R}} v_\varphi = v_\varphi(0) = \frac{1}{6}. \tag{3.3}$$

Note $\varphi - \partial_x^2 \varphi = 2\delta$. Here, δ denotes the Dirac distribution. For simplicity, we abuse notation by writing integrals instead of the H^{-1}/H^1 duality pairing. Hence we have

$$E_2(\varphi) = \|\varphi\|_X^2 = \int_{\mathbb{R}} (1 - \partial_x^2)\varphi \ (4 - \partial_x^2)^{-1}\varphi \ dx$$
$$= 2 \int_{\mathbb{R}} \delta(x)(4 - \partial_x^2)^{-1}\varphi(x) \ dx = 2v_\varphi(0) = \frac{1}{3}. \tag{3.4}$$

Lemma 3.1 *For any $u \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$, we have*

$$E_2(u) - E_2(\varphi) = \|u - \varphi(\cdot - \xi)\|_X^2 + 4 \left(v_u(\xi) - v_\varphi(0) \right),$$

where $v_u = (4 - \partial_x^2)^{-1} u$.

Proof. This can be done by a simple calculation. To see this, we have

$$\|u - \varphi(\cdot - \xi)\|_X^2 = \|u\|_X^2 + \|\varphi\|_X^2 - 2 \int_{\mathbb{R}} (1 - \partial_x^2)\varphi(x - \xi)(4 - \partial_x^2)^{-1} u(x) dx$$
$$= \|u\|_X^2 + \|\varphi\|_X^2 - 4 \int_{\mathbb{R}} \delta(x - \xi)(4 - \partial_x^2) - 1 u(x) dx$$
$$= \|u\|_X^2 + \|\varphi\|_X^2 - 4v_u(\xi) = \|u\|_X^2 - \|\varphi\|_X^2 + 2\|\varphi\|_X^2 - 4v_u(\xi)$$
$$= E_2(u) - E_2(\varphi) + 4 \left(v_\varphi(0) - v_u(\xi) \right).$$

where use has been made of integration by parts and the fact that $E_2(\varphi) = \|\varphi\|_X^2 = 2v_\varphi(0)$. This completes the proof of the lemma. $\square$

In the next two lemmas, we establish two formulas related the critical values of v_u to the two invariants $E_2(u)$ and $E_3(u)$. Consider a function $0 \neq u \in H^s, s > 3/2$ and $u \geq 0$. Then $0 < v_u = \left(4 - \partial_x^2\right)^{-1} u \in H^{s+2} \subset C^2$. Since v_u is positive and decays at infinity, it must have n points $\{\xi_i\}_{i=1}^n$ with local maximal values and $n - 1$ points $\{\eta_i\}_{i=1}^{n-1}$ with local minimal values for some integer $n \geq 1$. We arrange these critical points in their order by

$$-\infty < \xi_1 < \eta_1 < \xi_2 < \eta_2 < ... < \xi_{i-1} < \eta_{i-1} < \xi_i < \eta_i < ... < \eta_{n-1} < \xi_n < +\infty.$$

Let

$$v_u(\xi_i) = M_i, \ 1 \leq i \leq n \quad \text{and} \quad v_u(\eta_i) = m_i, \ 1 \leq i \leq n - 1. \tag{3.5}$$

Here, we assume $n < +\infty$, that is, there are a finite number of minima and maxima of v_u. In the case when there are infinitely many maxima and minima, the proofs below can be modified simply by changing the finite sums to infinite sums.

Lemma 3.2 *Let $0 \neq u \in H^s, s > 3/2$ and $u \geq 0$. By the above notations, define the function g by*

$$g(x) = \begin{cases} 2v_u + \partial_x^2 v_u - 3\partial_x v_u, & \eta_{i-1} < x < \xi_i, \\ 2v_u + \partial_x^2 v_u + 3\partial_x v_u, & \xi_i < x < \eta_i, \end{cases} \quad 1 \leq i \leq n. \tag{3.6}$$

with $\eta_0 = -\infty$ and $\eta_n = +\infty$. Then we have

$$\int_{\mathbb{R}} g^2(x)dx = E_2(u) - 12 \left(\sum_{i=1}^n M_i^2 - \sum_{i=1}^{n-1} m_i^2 \right). \tag{3.7}$$

Proof. To simplify notations, we use v for v_u below. Then $u = 4v - \partial_x^2 v$. First, we note that

$$E_2(u) = \int_{\mathbb{R}} \left[(1 - \partial_x^2)u \right] vdx = \int_{\mathbb{R}} (uv + \partial_x u \partial_x v)\, dx$$

$$= \int \left\{ \left(4v - \partial_x^2 v \right) v + \left[\left(4 - \partial_x^2 \right) \partial_x v \right] \partial_x v \right\} dx$$

$$= \int \left[4v^2 + 5\left(\partial_x v \right)^2 + \left(\partial_x^2 v \right)^2 \right] dx.$$

To show (3.7), we evaluate the integral of g^2 on each interval $[\eta_{i-1}, \eta_i]$, $1 \leq i \leq n$. We have

$$\int_{\eta_{i-1}}^{\eta_i} g^2(x)dx = \int_{\eta_{i-1}}^{\xi_i} (2v + \partial_{xx}v - 3\partial_x v)^2 \, dx + \int_{\xi_i}^{\eta_i} (2v + \partial_{xx}v + 3\partial_x v)^2 \, dx$$

$$= I + II.$$

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To estimate the first term, by integration by parts, we have

$$
I = \int_{\eta_{i-1}}^{\xi_i} \left(4v^2 + (\partial_{xx}v)^2 + 9\left(\partial_x v\right)^2 + 4v\partial_{xx}v - 12v\partial_x v - 6\partial_{xx}v\partial_x v \right) dx
$$

$$
= \int_{\eta_{i-1}}^{\xi_i} \left(4v^2 + (\partial_{xx}v)^2 + 5\left(\partial_x v\right)^2 \right) dx - 6\left(v\left(\xi_i\right)^2 - v\left(\eta_{i-1}\right)^2 \right),
$$

where use has been made of the fact that $\partial_x v\left(\xi_i\right) = \partial_x v\left(\eta_{i-1}\right) = 0$. Similarly,

$$
II = \int_{\xi_i}^{\eta_i} \left(4v^2 + \left(\partial_x^2 v\right)^2 + 5\left(\partial_x v\right)^2 \right) dx + 6\left(v\left(\eta_i\right)^2 - v\left(\xi_i\right)^2 \right).
$$

So

$$
\int_{\eta_{i-1}}^{\eta_i} g^2(x)dx = \int_{\eta_{i-1}}^{\eta_i} \left(4v^2 + \left(\partial_x^2 v\right)^2 + 5\left(\partial_x v\right)^2 \right) dx - 12v\left(\xi_i\right)^2 + 6v\left(\eta_{i-1}\right)^2 + 6v\left(\eta_i\right)^2
$$

and

$$
\int_{\mathbb{R}} g^2(x)dx = \int_{\mathbb{R}} \left(4v^2 + \left(\partial_x^2 v\right)^2 + 5\left(\partial_x v\right)^2 \right) dx - \sum_{i=1}^{n} \left(12v\left(\xi_i\right)^2 - 6v\left(\eta_{i-1}\right)^2 - 6v\left(\eta_i\right)^2 \right)
$$

$$
= E_2\left(u\right) - 12\left(\sum_{i=1}^{n} M_i^2 - \sum_{i=1}^{n-1} m_i^2 \right),
$$

where use has been made of the fact that $v\left(\eta_0\right) = v\left(\eta_n\right) = 0$ and the notations in (3.5). $\square$

Lemma 3.3 *With the same assumptions and notations in Lemma 3.2. Define the function h by*

$$
h(x) = \left\{ \begin{array}{ll} -\partial_x^2 v_u - 6\partial_x v_u + 16v_u, & \eta_{i-1} < x < \xi_i, \\ -\partial_x^2 v_u + 6\partial_x v_u + 16v_u, & \xi_i < x < \eta_i, \end{array} \right. \quad 1 \leq i \leq n. \qquad (3.8)
$$

with $\eta_0 = -\infty$ and $\eta_n = +\infty$. Then we have

$$
\int_{\mathbb{R}} h(x)g^2(x)dx = E_3(u) - 144\left(\sum_{i=1}^{n} M_i^3 - \sum_{i=1}^{n-1} m_i^3 \right). \qquad (3.9)
$$

Proof. We still use v for v_u. First, note that

$$
E_3(u) = \int_{\mathbb{R}} \left(4v - \partial_{xx}v \right)^3 dx = \int_{\mathbb{R}} \left[-\left(\partial_{xx}v\right)^3 + 12\left(\partial_{xx}v\right)^2 v - 48v^2\partial_{xx}v + 64v^3 \right] dx.
$$

To show (3.9), we evaluate the integral of $h(x)g^2(x)$ on each interval $[\eta_{i-1}, \eta_i]$, $1 \leq i \leq n$. We have

$$
\int_{\eta_{i-1}}^{\eta_i} h\left(x\right) g^2(x)dx = \int_{\eta_{i-1}}^{\xi_i} \left(-\partial_{xx}v - 6\partial_x v + 16v\right)\left(2v + \partial_{xx}v - 3\partial_x v\right)^2 dx
$$

$$
+ \int_{\xi_i}^{\eta_i} \left(-\partial_{xx}v + 6\partial_x v + 16v\right)\left(2v + \partial_{xx}v + 3\partial_x v\right)^2 dx
$$

$$
= I + II.
$$

It is found that the first term

$$
\begin{aligned}
I &= \int_{\eta_{i-1}}^{\xi_i} \{-\left(\partial_{xx}v\right)^3 + 12\left(\partial_{xx}v\right)^2 v + 27\partial_{xx}v\left(\partial_x v\right)^2 - 108v\partial_{xx}v\partial_x v + 60v^2\partial_{xx}v \\
&\qquad - 54\left(\partial_x v\right)^3 + 216\left(\partial_x v\right)^2 v - 216v^2\partial_x v + 64v^3\}dx \\
&= \int_{\eta_{i-1}}^{\xi_i} \{-\left(\partial_{xx}v\right)^3 + 12\left(\partial_{xx}v\right)^2 v + 54\left(\partial_x v\right)^3 + 60v^2\partial_{xx}v - 54\left(\partial_x v\right)^3 \\
&\qquad - 108v^2\partial_{xx}v + 64v^3\}dx - 72\left(v\left(\xi_i\right)^3 - v\left(\eta_{i-1}\right)^3\right) \\
&= \int_{\eta_{i-1}}^{\xi_i} \left[-\left(\partial_{xx}v\right)^3 + 12\left(\partial_{xx}v\right)^2 v - 48v^2\partial_{xx}v + 64v^3\right] dx - 72\left(v\left(\xi_i\right)^3 - v\left(\eta_{i-1}\right)^3\right),
\end{aligned}
$$

where use has been made of the following integral identities due to integration by parts and $\partial_x v\left(\xi_i\right) = \partial_x v\left(\eta_{i-1}\right) = 0$,

$$
\int_{\eta_{i-1}}^{\xi_i} \partial_{xx}v\left(\partial_x v\right)^2 dx = \frac{1}{3}\int_{\eta_{i-1}}^{\xi_i} \partial_x\left(\left(\partial_x v\right)^3\right) dx = 0
$$

$$
\int_{\eta_{i-1}}^{\xi_i} v\partial_{xx}v\partial_x v\, dx = \int_{\eta_{i-1}}^{\xi_i} v\partial_x\left(\frac{1}{2}\left(\partial_x v\right)^2\right) dx = -\frac{1}{2}\int_{\eta_{i-1}}^{\xi_i}\left(\partial_x v\right)^3 dx,
$$

$$
\int_{\eta_{i-1}}^{\xi_i}\left(\partial_x v\right)^2 v\, dx = \int_{\eta_{i-1}}^{\xi_i} \partial_x v\partial_x\left(\frac{1}{2}v^2\right) dx = -\frac{1}{2}\int_{\eta_{i-1}}^{\xi_i} v^2\partial_{xx}v dx,
$$

$$
\int_{\eta_{i-1}}^{\xi_i} v^2\partial_x v dx = \int_{\eta_{i-1}}^{\xi_i}\frac{1}{3}\partial_x\left(v^3\right) dx = \frac{1}{3}\left(v\left(\xi_i\right)^3 - v\left(\eta_{i-1}\right)^3\right).
$$

Similarly,

$$
II = \int_{\xi_i}^{\eta_i}\left[-\left(\partial_{xx}v\right)^3 + 12\left(\partial_{xx}v\right)^2 v - 48v^2\partial_{xx}v + 64v^3\right] dx + 72\left(v\left(\eta_i\right)^3 - v\left(\xi_i\right)^3\right)
$$

and thus

$$
\begin{aligned}
\int_{\eta_{i-1}}^{\eta_i} h\left(x\right) g^2(x)dx &= \int_{\eta_{i-1}}^{\eta_i}\left[-\left(\partial_{xx}v\right)^3 + 12\left(\partial_{xx}v\right)^2 v - 48v^2\partial_{xx}v + 64v^3\right] dx \\
&\qquad - 144v\left(\xi_i\right)^3 + 72\left(v\left(\eta_{i-1}\right)^3 + v\left(\eta_i\right)^3\right).
\end{aligned}
$$

By adding up the above integral from 1 to n, we get

$$
\begin{aligned}
\int_{\mathbb{R}} h(x)g^2(x)dx &= \int_{\mathbb{R}}\left[-\left(\partial_{xx}v\right)^3 + 12\left(\partial_{xx}v\right)^2 v - 48v^2\partial_{xx}v + 64v^3\right] dx \\
&\qquad - 144\sum_{i=1}^{n} v\left(\xi_i\right)^3 + 72\sum_{i=1}^{n}\left(v\left(\eta_{i-1}\right)^3 + v\left(\eta_i\right)^3\right) \\
&= E_3(u) - 144\left(\sum_{i=1}^{n} M_i^3 - \sum_{i=1}^{n-1} m_i^3\right).
\end{aligned}
$$

□

Without changing the integral identities (3.7) and (3.9), we can rearrange M_i and m_i in the order:

$$M_1 \geq M_2 \cdots \geq M_n \geq 0, \quad m_1 \geq \cdots \geq m_{n-1} \geq 0.$$

Moreover, since each local minimum is less than the neighboring local maximum, we have $M_i \geq m_{i-1}$ $(2 \leq i \leq n)$. The following two elementary inequalities are needed in the later proofs.

Lemma 3.4 *For any $n \geq 2$, assume $\{M_i\}_{i=1}^{n}$ and $\{m_i\}_{i=1}^{n-1}$ are $2n-1$ numbers satisfy*

$$M_1 \geq M_2 \cdots \geq M_n \geq 0, \quad m_1 \geq \cdots \geq m_{n-1} \geq 0$$

and $M_i \geq m_{i-1}$ $(2 \leq i \leq n)$. Then

(i)

$$\sum_{i=2}^{n} \left(M_i^3 - m_{i-1}^3\right) \leq \frac{3}{2}M_1 \sum_{i=2}^{n} \left(M_i^2 - m_{i-1}^2\right). \tag{3.10}$$

(ii)

$$\left(M_1^2 + \sum_{2}^{n} \left(M_i^2 - m_{i-1}^2\right)\right)^{\frac{1}{2}} \geq \left(M_1^3 + \sum_{2}^{n} \left(M_i^3 - m_{i-1}^3\right)\right)^{\frac{1}{3}}. \tag{3.11}$$

Proof. (i) For any $2 \leq i \leq n$, we have

$$\left(M_i^3 - m_{i-1}^3\right) - \frac{3}{2}M_1 \left(M_i^2 - m_{i-1}^2\right)$$

$$= -\frac{1}{2} \left(M_i - m_{i-1}\right) \left(3M_1 m_{i-1} + 3M_1 M_i - 2M_i m_{i-1} - 2M_i^2 - 2m_{i-1}^2\right) \leq 0$$

since $M_1 \geq M_i \geq m_{i-1}$. This implies that desired result in (3.10).

(ii) Denote

$$A_n = \left(M_1^2 + \sum_{2}^{n} \left(M_i^2 - m_{i-1}^2\right)\right)^{\frac{1}{2}} \quad \text{and} \quad B_n = \left(M_1^3 + \sum_{2}^{n} \left(M_i^3 - m_{i-1}^3\right)\right)^{\frac{1}{3}}. \tag{3.12}$$

We want to show $A_n \geq B_n$ $(n \geq 2)$ by induction. For the case of $n = 2$, it is equivalent to show that

$$\left(M_1^2 + M_2^2 - m_1^2\right)^3 - \left(M_1^3 + M_2^3 - m_1^3\right)^2 \geq 0$$

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if $M_1 \geq M_2 \geq m_1 \geq 0$. We have

$$\left(M_1^2 + M_2^2 - m_1^2\right)^3 - \left(M_1^3 + M_2^3 - m_1^3\right)^2$$
$$= 3M_1^4 M_2^2 - 3M_1^4 m_1^2 - 2M_1^3 M_2^3 + 2M_1^3 m_1^3 + 3M_1^2 M_2^4$$
$$- 6M_1^2 M_2^2 m_1^2 + 3M_1^2 m_1^4 - 3M_2^4 m_1^2 + 2M_2^3 m_1^3 + 3M_2^2 m_1^4 - 2m_1^6$$
$$= (M_2 - m_1)(M_1 - m_1)\{3M_1 M_2^3 + 3M_1^3 M_2 - 2M_1 m_1^3 + 3M_1^3 m_1$$
$$- 2M_2 m_1^3 + 3M_2^3 m_1 - 2m_1^4 - 2M_1^2 M_2^2 + M_1^2 m_1^2 + M_2^2 m_1^2$$
$$- 2M_1 M_2 m_1^2 + M_1 M_2^2 m_1 + M_1^2 M_2 m_1\},$$

which is obviously nonnegative by the assumption $M_1 \geq M_2 \geq m_1 \geq 0$. Assume the inequality $A_n \geq B_n$ is true for $n \leq k$ ($k \geq 2$). Our goal is to deduce $A_{k+1} \geq B_{k+1}$. Since $A_k \geq M_1 \geq M_{k+1} \geq m_k$, we have

$$A_{k+1}^6 = \left[M_1^2 + \sum_2^{k+1}\left(M_i^2 - m_{i-1}^2\right)\right]^3$$
$$= \left(A_k^2 + M_{k+1}^2 - m_k^2\right)^3 \geq \left(A_k^3 + M_{k+1}^3 - m_k^3\right)^2 \text{ (by the } n = 2 \text{ inequality)}$$
$$\geq \left(B_k^3 + M_{k+1}^3 - m_k^3\right)^2 \text{ (since } A_k \geq B_k \text{ by the induction assumption)}$$
$$= B_{k+1}^6.$$

Thus $A_{k+1} \geq B_{k+1}$ and $A_n \geq B_n$ is true for any $n \geq 2$. $\square$

The following lemma is crucial in the proof of stability of the peakons.

Lemma 3.5 *Assume $u_0 \in H^s, s > 3/2$ and $y_0 \geq 0$. Let $M_1 = v_u(t, \xi_1) = \max_{x \in \mathbb{R}}\{v(t,x)\}$ Then for $t \geq 0$,*

$$E_3(u) - 144B_n^3 \leq 18M_1\left(E_2(u) - 12A_n^2\right) \tag{3.13}$$

where u is the global solution of equation (1.5) with initial value u_0, $v_u = (4 - \partial_x^2)^{-1}u$, and A_n and B_n are defined in (3.12).

Proof. First, by Lemma 2.8 the global solution u of equation (1.5) satisfies $u(t,x) \geq 0$ and $y(t,x) = u - \partial_x^2 u \geq 0$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. We now claim that $h \leq 18v$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. To see this, we rewrite the expression of h as

$$h(x) = \begin{cases} -\left(\partial_x^2 + 3\partial_x + 12\right)v - 3\partial_x v + 18v, & \eta_{i-1} < x < \xi_i, \\ -\left(\partial_x^2 - 3\partial_x + 2\right)v + 3\partial_x v + 18v, & \xi_i < x < \eta_i, \end{cases} \quad 1 \leq i \leq n.$$

If $\eta_{i-1} < x < \xi_i$, $1 \leq i \leq n$, then $v_x > 0$. On the other hand, it follows from Lemma 2.10 that

$$-\left(\partial_x^2 + 3\partial_x + 2\right)v = -(2 + \partial_x)(4 - \partial_x^2)^{-1}(1 + \partial_x)u \leq 0.$$

Hence

$$-\left(\partial_x^2 + 3\partial_x + 2\right)v - 3\partial_x v + 18v \leq 18v. \tag{3.14}$$

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A similar argument also shows that for $\xi_i < x < \eta_i,\ 1 \le i \le n,\ \partial_x v < 0$ and

$$- \left(\partial_x^2 - 3\partial_x + 2\right) v + 3\partial_x v + 18v \tag{3.15}$$
$$= -(2 - \partial_x)(4 - \partial_x^2)^{-1}(1 - \partial_x)u + 3\partial_x v + 18v \le 18v.$$

The combination of (3.14) and (3.15) yields

$$h \le 18v \le 18 \max v = 18M_1, \quad \forall\, (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

By the notation in (3.12), the integral identities (3.7) and (3.9) become

$$\int_{\mathbb{R}} g^2(x)dx = E_2(u) - 12A_n^2$$

and

$$\int_{\mathbb{R}} h(x)g^2(x)dx = E_3(u) - 144B_n^3.$$

Note that when $n = 1,\ A_1 = B_1 = M_1$. Relating the above integrals, we get

$$E_3(u) - 144B_n^3 \le 18M_1(E_2(u) - 12A_n^2).$$

$\square$

Lemma 3.6 *Assume* $u \in H^s,\ s > 3/2$ *and* $y \ge 0$. *Let*

$$M_1 = v_u(t, \xi_1) = \max_{x \in \mathbb{R}}\{v(t, x)\}$$

and

$$A_n = M_1^2 + \sum_{i=2}^{n} \left(M_i^2 - m_{i-1}^2\right)$$

where M_i *and* m_{i-1} $(2 \le i \le n)$ *are local maxima and minima of* v_u. *If* $|E_2(u) - E_2(\varphi)| \le \delta$ *and* $|E_3(u) - E_3(\varphi)| \le \delta$ *with* $0 < \delta < 1$, *then*

    (i)

$$\left| M_1 - \frac{1}{6} \right| < \sqrt{\delta}, \tag{3.16}$$

recalling that $v_\varphi(0) = \frac{1}{6} = \max v_\varphi$,

    (ii)

$$\left| A_n - \frac{1}{6} \right| < \sqrt{\delta}, \tag{3.17}$$

and

    (iii)

$$\sum_{i=2}^{n} \left(M_i^2 - m_{i-1}^2\right) < \frac{4}{3}\sqrt{\delta}. \tag{3.18}$$

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Proof. To obtain (i), we first claim that

$$M_1^3 - \frac{1}{4}E_2(u)M_1 + \frac{1}{72}E_3(u) \le 0. \tag{3.19}$$

In the case when M_1 is the only local maximum of v_u, we have $n = 1$, $A_1 = B_1 = M_1$ and (3.19) follows directly from (3.13). When $n \ge 2$, in view of (3.13) and inequality (3.10) in Lemma 3.4 (i), there appears the relation

$$M_1^3 - \frac{1}{4}E_2(u)M_1 + \frac{1}{72}E_3(u)$$

$$\le 144 \left(\sum_{i=2}^{n} \left(M_i^3 - m_{i-1}^3 \right) - \frac{3}{2}M_1 \sum_{i=2}^{n} \left(M_i^2 - m_{i-1}^2 \right) \right) \le 0.$$

Define the cubic polynomial P by

$$P(y) = y^3 - \frac{1}{4}E_2(u)y + \frac{1}{72}E_3(u). \tag{3.20}$$

For the peakon solution, $E_2(\varphi) = \frac{1}{3}$ and $E_3(\varphi) = \frac{2}{3}$, the above polynomial becomes

$$P_0(y) = y^3 - \frac{1}{12}y + \frac{1}{108} = \left(y - \frac{1}{6} \right)^2 \left(y + \frac{1}{3} \right). \tag{3.21}$$

Since

$$P_0(M_1) = P(M_1) + \frac{1}{4} \left(E_2(u) - E_2(\varphi) \right) M_1 - \frac{1}{72} \left(E_3(u) - E_3(\varphi) \right),$$

and $P(M_1) \le 0$ by (3.19) , it follows that

$$\left(M_1 - \frac{1}{6} \right)^2 \le \frac{3}{4} \left(E_2(u) - E_2(\varphi) \right) M_1 - \frac{1}{24} \left(E_3(u) - E_3(\varphi) \right). \tag{3.22}$$

On the other hand, observing $E_2(u) - 12A_n^2 \ge 0$, we have

$$0 < M_1 \le A_n \le \sqrt{E_2(u)/12} \le \sqrt{(1/3 + \delta)/12} < \frac{1}{3} \tag{3.23}$$

when $\delta < 1$. It is then inferred from (3.22) that

$$\left| M_1 - \frac{1}{6} \right| \le \sqrt{\frac{1}{4} |E_2(u) - E_2(\varphi)| + \frac{1}{24} |E_3(u) - E_3(\varphi)|} < \sqrt{\delta}.$$

We now prove claim (ii). When $n = 1$, $A_1 = M_1$ and it is reduced to (i). When $n \ge 2$, it is thereby inferred from (3.13) that

$$A_n^3 - \frac{1}{4}E_2(u)A_n + \frac{1}{72}E_3(u) \le 0, \tag{3.24}$$

due to $0 \le M_1 \le A_n$ and $0 \le B_n \le A_n$ by Lemma 3.4 (ii). In consequence, (3.17) follows from the same argument as in part (i).

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(iii) can be obtained from (i) and (ii). In fact, combining (i) and (ii), we have

$$2\sqrt{\delta} \geq A_n - M_1 = \frac{\sum_2^n \left(M_i^2 - m_{i-1}^2\right)}{A_n + M_1},$$

which implies (3.18) by using (3.23). $\square$

Proof of Theorem 1. Let $u \in C([0, \infty), H^s)$, $s > 3/2$ be the solution of (1.5) with initial data $u(0) = u_0$. Since E_2 and E_3 are both conserved by the evolution equation (1.5), we have

$$E_2(u(t, \cdot)) = E_2(u_0) \quad \text{and} \quad E_3(u(t, \cdot) = E_3(u_0), \quad \forall t \geq 0. \tag{3.25}$$

Since $\|u_0 - \varphi\|_X < \varepsilon$, we obtain

$$|E_2(u_0) - E_2(\varphi)| = |(\|u_0\|_X - \|\varphi\|_X)(\|u_0\|_X + \|\varphi\|_X)|$$

$$\leq \varepsilon \left(2\|\varphi\|_X + \varepsilon\right) = \varepsilon \left(\frac{2}{\sqrt{3}} + \varepsilon\right) < 2\varepsilon,$$

under the assumption $\varepsilon < \frac{1}{2}$. In view of (3.25), the assumptions of Lemma 3.6 are satisfied for $u(t, \cdot)$ and $\delta = 2\varepsilon$. It is then inferred that

$$\left|v_u(t, \xi_1(t)) - \frac{1}{6}\right| \leq \sqrt{2\varepsilon}, \quad \forall t \geq 0. \tag{3.26}$$

By (3.25) and Lemma 3.1, we have

$$\|u(t, \cdot) - \varphi(\cdot - \xi_1(t))\|_X^2 = E_2(u_0) - E_2(\varphi) + 4(v_\varphi(0) - v_u(t, \xi_1(t))), \quad \forall t \geq 0.$$

Combining the above estimates yields

$$\|u(t, \cdot) - \varphi(\cdot - \xi_1(t))\|_X \leq \sqrt{2\varepsilon + 4\sqrt{2\varepsilon}} < 3\varepsilon^{1/4}. \quad \forall t \geq 0.$$

Estimates (1.9) and (1.10) then follow directly from Lemma 3.6 (ii) and (iii). This completes the proof of Theorem 1. $\square$

Remark 2 *We make several comments.*

(1) By (3.23), $M_1 = \max v_u \leq \sqrt{E_2(u)/12}$. For peakons $c\varphi$, we have $\max v_{c\varphi} = \sqrt{E_2(c\varphi)/12} = \frac{1}{6}c$. So among all waves of a fixed energy E_2, the peakon is tallest in terms of v_u.

(2) In our proof, we use inequality (3.22) to get estimates (3.16) and (3.18) more directly, compared with the argument in [11] by analyzing the root structure of the polynomial $P(y)$. Moreover, it implies that the peakons are energy minimizers with a fixed invariant E_3, which explains their stability. Indeed, if $E_3(u) = E_3(\varphi)$, it follows from (3.22) that $E_2(u) \geq E_2(\varphi)$. The same remark also applies to the CH equation and shows that the CH-peakons are energy minima with fixed F_3.

(3) Compared with [11], our construction of the integral relations (3.7) and (3.9) is more delicate. It not only is required in our current case, but also

provides us more information about stability via (1.10). For the CH equation, even if the orbital stability is proved by a simpler construction [11], our approach can also give the additional stability information. More specifically, for the CH equation (1.3) with $y_0 \geq 0$, by refining the integrals of [11, Lemma 2] to each monotonic interval of u, one can obtain

$$F_3\left(u\right) - \frac{4}{3}B_n^3 \leq M_1\left(F_2\left(u\right) - 2A_n^2\right),$$

where F_2 and F_3 are defined in (1.7), and A_n and B_n in (3.12) with M_i and m_i being the maxima and minima of u, respectively. Hence, estimate (1.10) may be obtained by following the proof of Lemma 3.6.

Acknowledgements

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