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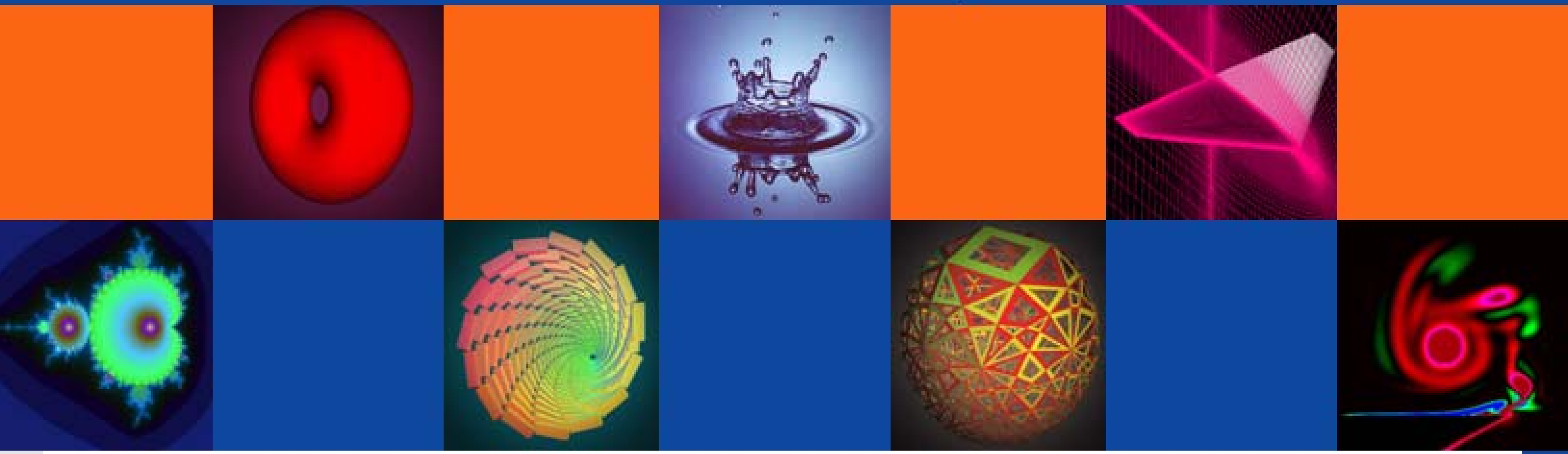

Error Analysis for Weighted Higher Order Compact Finite Difference Scheme

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Error Analysis for Weighted Higher Order Compact Finite Difference Scheme

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Abstract: In this paper, we analyze a type of finite-difference schemes, a newly developed Weighted Compact Scheme (WCS) developed by L. Jiang L, H. Shan, C. Liu, [Weight Compact Scheme for Shock Capturing, *International Journal of Computational Fluid Dynamics*, 15, pp.147-155 (2001)], in terms of order of accuracy and numerical dissipation and dispersion. Further results of numerical implementation have indicated their effectiveness in approximating shock discontinuities. The scheme is shown to be of high order of accuracy in regions where the solutions are smooth; while in regions of shock, they are of lower orders but are capable to overcome numerical oscillations with the help of weights. The scheme is compactable and complements with the famous WENO scheme. Our analysis has shown the plausibility that the WCS and WENO schemes can work together to provide a numerical method to efficiently capture discontinuities as well as other small-scale features.

Keyword: Finite Difference Schemes, High Order Compact Schemes, WENO, Order of Accuracy, Numerical Methods for Partial Differential Equations.

2000 Mathematics Subject Classification. 65M06, 65M06, 35L67

1. Introduction.

The central issue of Computational Fluid Dynamics has been around the development of numerical methods for calculating flow fields that are governed in general by the Navier-Stokes system. One of the difficult problems with numerical solution is the capturing of shocks that are discontinuity or mathematical singularity in the partial differential equations. In these regions, there is no classical solution for the equation and bounded derivatives of the solutions do not exist. Only weak solutions in an integration form of the equations can be obtained. In the problems of high speed or turbulent flows, the shocks can be developed over the time course because the dominant mode of the flow, described by Euler equation, is non-linear and hyperbolic.

In the existing mathematical and computational physics literature, many finite difference schemes such as Godunov (1969) [1], Roe (1981) [2], MUSCL (Van Leer, 1979) [3], TVD (Harten, 1983) [4], ENO (Harten et al, 1987 [5]; Shu et al, 1988 [6], 1989 [7]) and WENO (Liu et al, 1994 [8]; Jiang et al, 1996 [9]), have achieved great success in capturing the shocks sharply. The basic idea proposed in ENO (Harten et al, 1987 [5]) and WENO (Jiang et al, 1996 [9]) schemes is to avoid the stencil containing a shock. ENO chooses the smoothest stencil from several candidates to calculate the derivatives. WENO controls the contributions of different stencils according to their smoothness. In this way, the derivative at a certain grid point, especially one near the shock, is dependent on a very limited number of grid points. The local dependency here is favorable for shock capturing and helps obtaining the non-oscillatory property. The success of ENO and WENO schemes indicates that the local dependency is critical for shock capturing.

Recent research directions further require methods with fine resolution for small length scales. Such feature is required in studying physics phenomena in flow transition and turbulence processes that are very sensitive to any artificial numerical dissipation. High order compact scheme (Lele, 1992 [10]; Visbal, 2002 [11]) is more appropriate for simulation in this area because it is central and non-dissipative with high order accuracy and high resolution. Due to the usage of derivatives, compact schemes usually give us a tri-diagonal or penta-diagonal system. Although the tri-diagonal matrix is sparse, the inverse of a tri-diagonal matrix is dense, which means the derivative at a certain grid point depends upon all the grid points along a grid line. The success of compact schemes indicates that the global dependency is very important for high resolution. However, the global dependency is not so applicable for shock capturing.

To balance the benefit and shortcoming of both compact scheme and WENO, Weighted Compact Scheme (WCS) was recently developed (Jiang et al, 2001 [12]) by introducing the idea of WENO scheme to the standard compact schemes which uses weights for nearby several candidates. In smooth regions where shocks are not present, the scheme is identical to standard compact scheme in [10-11]. The building block for each candidate is a Lagrange polynomial in WENO, but is Hermite in WCS. Therefore WCS achieves a higher accuracy with same stencil width. While in shock region, WCS tries to minimize its influence that the stencil containing a shock is still used with a smaller weight leading to the global dependency. The numerical tests indicated that WCS works fine in convection equation and Burger's equation [12].

The order of accuracy is an important feature for a finite difference scheme. For a higher order scheme, the requirement of grid number can be substantially less that that of a lower order. The order of accuracy is one of the measuring criterions we use for improving the numerical method. Through investigating the error terms, we can further have a clear understanding about the numerical dissipation and dispersion generated by the scheme.

In this note, we perform an error analysis for Weighted Compact Scheme. The analysis will provide a way to optimize the simulation results for a wide range of fluid problems. In section 2, we briefly introduce the scheme formulations, in Section 3, we study the error analysis; and in Section 4 we will illustrate numerical examples of WCS scheme in comparison with other schemes such as Compact Schemes and WENO scheme.

Our discussion only concerns with one-dimension scheme, and higher dimension cases will be reported in a forthcoming publication.

2. Finite Difference Scheme Formulations

For 1-D conservation law:

$$u_t(x,t) + F_x(u(x,t)) = 0, \qquad (2.1)$$

when a conservative approximation to the spatial derivative is applied, a semi-discrete conservative form of eq. (2.1) is described as follows:

$$\frac{du_j}{dt} = -\frac{1}{\Delta x}(\hat{F}_{j+(1/2)} - \hat{F}_{j-(1/2)}) \qquad (2.2)$$

where $F_j \equiv F(u(x_j)) = \frac{1}{\Delta x}\int_{x_j-\Delta x/2}^{x_j+\Delta x/2} \hat{F}(\xi)d\xi$ and then $(F_x)_j = -\frac{1}{\Delta x}(\hat{F}_{j+(1/2)} - \hat{F}_{j-(1/2)})$.

Note that F is the original function, but $\hat{F}$ is the flux defined by the above integration. Eq. (2.2) is an exact expression of eq. (2.1) but $\hat{F}$ is different from F. Let H be the primitive function of $\hat{F}$ defined as:

$$H(x_{j+(1/2)}) = \int_{-\infty}^{x_j+\Delta x/2} \hat{F}(\xi)d\xi = \sum_{i=-\infty}^{i=j} \int_{x_i-\Delta x/2}^{x_i+\Delta x/2} \hat{F}(\xi)d\xi = \Delta x \sum_{i=-\infty}^{j} f_i$$

(2.3)

H usually is to be calculated on a discrete data set. The numerical flux $\hat{F}$ at the cell interfaces is the derivative of its primitive function H. i.e.: $\hat{F}_{j+(1/2)} = H'_{j+(1/2)}$

(2.4)

Our study is closely related to WENO schemes, particularly its weight distribution. Therefore we start with the standard WENO setting.

WENO setting:

In order to get a second order approximation for $\hat{F}_{j-\frac{1}{2}}$, we can use three different candidates (Figure 1): $E_0 : F_{j-3}, F_{j-2}, F_{j-1}$; $E_1 : F_{j-2}, F_{j-1}, F_j$; $E_2 : F_{j-1}, F_j, F_{j+1}$

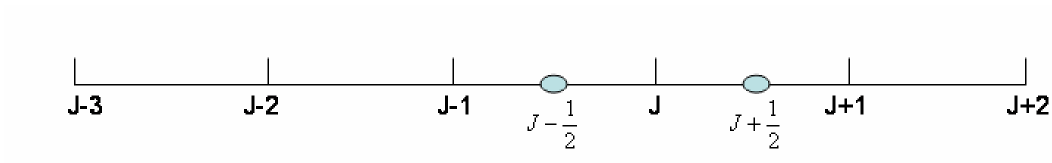


Figure 1. Fifth order WENO Scheme diagram

$$E_0 : p_{0,j-1/2}(x) = -\frac{F_{j-3} - 26F_{j-2} + F_{j-1}}{24} + \frac{F_{j-1} - F_{j-3}}{2h}(x - x_{j-2}) + \frac{F_{j-3} - 2F_{j-2} + F_{j-1}}{2h^2}(x - x_{j-2})^2,$$

$$E_1 : p_{1,j-1/2}(x) = -\frac{F_{j-2} - 26F_{j-1} + F_j}{24} + \frac{F_j - F_{j-2}}{2h}(x - x_{j-1}) + \frac{F_{j-2} - 2F_{j-1} + F_j}{2h^2}(x - x_{j-1})^2,$$

(2.5)

$$E_2 : p_{2,j-1/2}(x) = \frac{F_{j-1} - 26F_j + F_{j+1}}{24} + \frac{F_{j+1} - F_{j-1}}{2h}(x - x_j) + \frac{F_{j-1} - 2F_j + F_{j+1}}{2h^2}(x - x_j)^2,$$

and let $x = x_{j-\frac{1}{2}}$ in eq. (2.5). Appling a weighted average $E = C_0E_0 + C_1E_1 + C_2E_2$ with $C_0 = \frac{1}{10}, C_1 = \frac{6}{10}, C_2 = \frac{3}{10}$, we derive formula for $\hat{F}_{j-\frac{1}{2}}$ and also $\hat{F}_{j+\frac{1}{2}} = \hat{F}_{(j+1)-\frac{1}{2}}$.

Finally,

$$\frac{\partial F}{\partial x} = \frac{\hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}}}{\Delta x} \approx (-\frac{1}{30}F_{j-3} + \frac{1}{4}F_{j-2} - F_{j-1} + \frac{1}{3}F_j + \frac{1}{2}F_{j+1} - \frac{1}{20}F_{j+2}) / \Delta x. \quad (2.6)$$

Using Taylor expansion, we can verify that eq. (2.6) has a 5th order truncation error. Following [9], the proper weights are defined as $\omega_k = \frac{\gamma_k}{\sum_{i=0}^{2}\gamma_i}$, $\gamma_k = \frac{C_k}{(\varepsilon + IS_k)^p}$ with

$$IS_i = \int_{x_{j-1/2}}^{x_{j+1/2}} \sum_{k=1}^{2} [p_i(x)^{(k)}]^2 \Delta x^{2k-1} dx.$$

The 5th order WENO can be obtained by the new weighted average

$$\hat{F}_{j-1/2} \approx \omega_{0,j-1/2}(\frac{1}{3}F_{j-3} - \frac{7}{6}F_{j-2} + \frac{11}{6}F_{j-1}) + \omega_{1,j-1/2}(-\frac{1}{6}F_{j-2} + \frac{5}{6}F_{j-1} + \frac{1}{3}F_j)$$
$$+ \omega_{2,j-1/2}(\frac{1}{3}F_{j-1} + \frac{5}{6}F_j - \frac{1}{6}F_{j+1}),$$

$$\hat{F}_{j+1/2} \approx \omega_{0,j+1/2}(\frac{1}{3}F_{j-2} - \frac{7}{6}F_{j-1} + \frac{11}{6}F_j) + \omega_{1,j+1/2}(-\frac{1}{6}F_{j-1} + \frac{5}{6}F_j + \frac{1}{3}F_{j+1})$$
$$+ \omega_{2,j+1/2}(\frac{1}{3}F_j + \frac{5}{6}F_{j+1} - \frac{1}{6}F_{j+2}),$$

$$\frac{\partial F}{\partial x} \approx \frac{\hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}}}{\Delta x} = [-\frac{1}{3}\omega_{0,j-1/2}F_{j-3} + (\frac{7}{6}\omega_{0,j-1/2} + \frac{1}{3}\omega_{0,j+1/2} + \frac{1}{6}\omega_{1,j-1/2})F_{j-2}$$
$$+ (-\frac{11}{6}\omega_{0,j-1/2} - \frac{7}{6}\omega_{0,j+1/2} - \frac{5}{6}\omega_{1,j-1/2} - \frac{1}{6}\omega_{1,j+1/2} - \frac{1}{3}\omega_{2,j-1/2})F_{j-1}$$
$$+ (\frac{11}{6}\omega_{0,j+1/2} - \frac{1}{3}\omega_{1,j-1/2} + \frac{5}{6}\omega_{1,j+1/2} - \frac{5}{6}\omega_{2,j-1/2} + \frac{1}{3}\omega_{2,j+1/2})F_j \tag{2.7}$$
$$+ (\frac{1}{3}\omega_{1,j+1/2} + \frac{1}{6}\omega_{2,j-1/2} + \frac{5}{6}\omega_{2,j+1/2})F_{j+1} - \frac{1}{6}\omega_{2,j+1/2}F_{j+2}]/\Delta x.$$

Our weighted compact scheme goes to a different direction from the WENO scheme, however we will use the same weight function, as numerical experiments indicated that WENO weights provides most stability out of several choices, the theoretical base of such is under investigation.

WCS setting:
The high order compact scheme also uses three candidates (Figure 2):
$E_0 : H_{j-3/2}, H_{j-1/2}, H_{j+1/2}$, $E_1 : H_{j-1/2}, H_{j+1/2}, H_{j+3/2}$, and $E_2 : H_{j+1/2}, H_{j+3/2}, H_{j+5/2}$

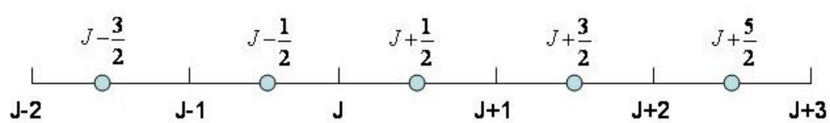


Figure 2. Sixth Order Compact Scheme diagram

$H_{j+1/2} = \sum_{i=0}^{j} F_i \Delta x$, and we introduce lower order scheme on each E_i according to [10-11]:

E_0: $\alpha_0 H'_{j-1/2} + H'_{j+1/2} = (-b_0 H_{j-3/2} - a_0 H_{j-1/2} + c_0 H_{j+1/2})/\Delta x$

E_1: $\alpha_1 H'_{j-1/2} + H'_{j+1/2} + \alpha_1 H'_{j+3/2} = a_1(H_{j+3/2} - H_{j-1/2})/\Delta x$

(2.8)

E_2: $H'_{j+1/2} + \alpha_2 H'_{j+3/2} = (b_2 H_{j+5/2} + a_2 H_{j+3/2} - c_2 H_{j+1/2})/\Delta x$

where $\alpha_0 = 2, \alpha_1 = \frac{1}{4}, \alpha_2 = 2, a_0 = 2, a_1 = \frac{3}{4}, a_2 = 2, b_0 = \frac{1}{2}, b_2 = \frac{1}{2}, c_0 = \frac{5}{2}, c_2 = \frac{5}{2}$.

The approximations E_0 and E_2 have third order of accuracy, but E_1 has fourth order of accuracy. Let $E = C_0 E_0 + C_1 E_1 + C_2 E_2$, $C_0 = C_2 = \frac{1}{18}$, $C_1 = \frac{8}{18}$, we derive in an explicit form:

$$\frac{1}{3}F'_{j-1} + F'_j + \frac{1}{3}F'_{j+1} \approx \frac{1}{h}(-\frac{1}{36}F_{j-2} - \frac{7}{9}F_{j-1} + \frac{7}{9}F_{j+1} + \frac{1}{36}F_{j+2}), \tag{2.9}$$

which is a standard sixth order compact scheme. Similarly, for Weighted Compact Scheme [12], using the same weights as in WENO scheme, we also derive

$$\hat{F}_{j+1/2} \approx \omega_{0,j+1/2}(\frac{1}{2}F_{j-1} + \frac{5}{2}F_j + 2\Delta x \bullet F_j^{'})/3 + \omega_{1,j+1/2}[\frac{3}{4}(F_{j+1} + F_j) + \frac{1}{4}\Delta x F_j^{'} - \frac{1}{4}\Delta x F_{j+1}^{'}]/1.5$$
$$+ \omega_{2,j+1/2}(\frac{1}{2}F_{j+2} + \frac{5}{2}F_{j+1} - 2\Delta x \bullet F_{j+1}^{'})/3$$

(2.10)

$$\hat{F}_{j-1/2} \approx \omega_{0,j-1/2}(\frac{1}{2}F_{j-2} + \frac{5}{2}F_{j-1} + 2\Delta x \bullet F_{j-1}^{'})/3 + \omega_{1,j-1/2}[\frac{3}{4}(F_j + F_{j-1}) + \frac{1}{4}\Delta x F_{j-1}^{'} - \frac{1}{4}\Delta x F_j^{'}]/1.5$$
$$+ \omega_{2,j-1/2}(\frac{1}{2}F_{j+1} + \frac{5}{2}F_j - 2\Delta x \bullet F_j^{'})/3$$

and then

$$F_j^{'} = \frac{\hat{F}_{j+1/2} - \hat{F}_{j-1/2}}{\Delta x} \approx \frac{1}{\Delta x}[\frac{1}{6}\omega_{2,j+1/2}F_{j+2} + (\frac{5}{6}\omega_{2,j+1/2} + \frac{1}{2}\omega_{1,j+1/2} - \frac{1}{6}\omega_{2,j-1/2})F_{j+1}$$
$$+ (\frac{1}{2}\omega_{1,j+1/2} + \frac{5}{6}\omega_{0,j+1/2} - \frac{1}{2}\omega_{1,j-1/2} - \frac{5}{6}\omega_{2,j-1/2})F_j$$
$$+ (\frac{1}{6}\omega_{0,j+1/2} - \frac{1}{2}\omega_{1,j-1/2} - \frac{5}{6}\omega_{0,j-1/2})F_{j-1} - \frac{1}{6}\omega_{0.j-1/2}F_{j-2}]$$
$$+ (-\frac{2}{3}\omega_{2,j+1/2} - \frac{1}{6}\omega_{1,j+1/2})F_{j+1}^{'} + (\frac{1}{6}\omega_{1,j+1/2} + \frac{2}{3}\omega_{0,j+1/2} + \frac{1}{6}\omega_{1,j-1/2} + \frac{2}{3}\omega_{2,j-1/2})F_j^{'}$$
$$+ (-\frac{1}{6}\omega_{1,j-1/2} - \frac{2}{3}\omega_{0,j-1/2})F_{j-1}^{'}.$$

(2.11)

3. Error Analysis

We now study the order of accuracy of WCS schemes.

Theorem 3. 1. *Assume eq. (2.1), a 1-D conservation law has a nonlinear function F that is differentiable up to 7th order and $u(x,t)$ is a generalized solution of eq. (2.1). Then given any fixed bounded domain $D \subset R^1$ and $0 < t \le T < \infty$ there exist $h_0 = h_0(u) > 0$ such that for any mesh size $\Delta x < h_0$, WCS scheme in eq. (2.10) gives 6th order accuracy in smooth regions of $u(x,t)$, i.e. $F_x - (\hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}})/\Delta x = O(\Delta x^6)$, and they achieve essentially non-oscillatory property.*

Proof: For the convenience of analysis, we first change eq. (2.8) to explicit forms. Note that:

$$F_j^{'} = \frac{\hat{F}_{j+1/2} - \hat{F}_{j-1/2}}{\Delta x} = \frac{H_{j+1/2}^{'} - H_{j-1/2}^{'}}{\Delta x} \qquad (2.12)$$

and

$$H'_{j-1/2} = H'_{j+1/2} - \Delta x \bullet F'_j,\ H'_{j+3/2} = H'_{j+1/2} + \Delta x \bullet F'_{j+1} \quad (2.13)$$

Substitute eq. (2.13) to the above implicit equations, we will get explicit formula:

E_0 : $H'_{j+1/2} \approx [(-b_0 H_{j-3/2} - a_0 H_{j-1/2} + c_0 H_{j+1/2}) / \Delta x + \alpha_0 \Delta x \bullet F'_j]/(1+\alpha_0)$

E_1 : $H'_{j+1/2} \approx [a_1 (H_{j+3/2} - H_{j-1/2}) / \Delta x + \alpha_1 \Delta x \bullet F'_j - \alpha_1 \Delta x \bullet F'_{j+1}]/(1+2\alpha_1)$

E_2 : $H'_{j+1/2} \approx [(b_2 H_{j+5/2} + a_2 H_{j+3/2} - c_2 H_{j+1/2}) / \Delta x - \alpha_2 \Delta x \bullet F'_{j+1}]/(1+\alpha_2)$

or equivalently

E_0 : $\hat{F}_{j+1/2} \approx [\frac{1}{2} F_{j-1} + \frac{5}{2} F_j + 2\Delta x \bullet F'_j]/3$

E_1 : $\hat{F}_{j+1/2} \approx [\frac{3}{4}(F_{j+1} + F_j) + \frac{1}{4}\Delta x \bullet F'_j - \frac{1}{4}\Delta x \bullet F'_{j+1}]/1.5 \quad (2.14)$

E_2 : $\hat{F}_{j+1/2} \approx [\frac{1}{2} F_{j+2} + \frac{5}{2} F_{j+1} - 2\Delta x \bullet F'_{j+1}]/3$.

Similarly, we have:

E_0 : $\hat{F}_{j-1/2} \approx [\frac{1}{2} F_{j-2} + \frac{5}{2} F_{j-1} + 2\Delta x \bullet F'_{j-1}]/3$

E_1 : $\hat{F}_{j-1/2} \approx [\frac{3}{4}(F_j + F_{j-1}) + \frac{1}{4}\Delta x \bullet F'_{j-1} - \frac{1}{4}\Delta x \bullet F'_j]/1.5$

E_2 : $\hat{F}_{j-1/2} \approx [\frac{1}{2} F_{j+1} + \frac{5}{2} F_j - 2\Delta x \bullet F'_j]/3$

All WCS implicit candidates become explicit and we can then proceed with analysis.

Non-bias weight: let take a look at candidate E_0

$$F'_j = \frac{\hat{F}_{j+1/2} - \hat{F}_{j-1/2}}{\Delta x} \approx [\frac{1}{2\Delta x}(-F_{j-2} - 4F_{j-1} + 5F_j) + 2F'_j - 2F'_{j-1}]/3.$$

Thus we derive in E_0 :

$$E_0\text{:} \quad 2F'_{j-1} + F'_j \approx \frac{1}{\Delta x}(-\frac{1}{2} F_{j-2} - 2F_{j-1} + \frac{5}{2} F_j).$$

Similarly in E_1 and E_2, we derive:

$$E_1\text{:} \quad \frac{1}{4} F'_{j-1} + F'_j + \frac{1}{4} F'_{j+1} \approx \frac{1}{\Delta x}(-\frac{3}{4} F_{j-1} + \frac{3}{4} F_{j+1}),$$

$$E_2\text{:} \quad 2F'_{j+1} + F'_j \approx \frac{1}{\Delta x}(-\frac{5}{2} F_j + 2F_{j+1} + \frac{1}{2} F_{j+2}).$$

Let us take a weighted average of 3 stencils,

$E = C_0 E_0 + C_1 E_1 + C_2 E_2$, $C_0 = C_2 = \frac{1}{18}$ *and* $C_1 = \frac{8}{18}$, then we derive

$$\frac{1}{3}F'_{j-1} + F'_j + \frac{1}{3}F'_{j+1} \approx \frac{1}{\Delta x}\left(-\frac{1}{36}F_{j-2} - \frac{7}{9}F_{j-1} + \frac{7}{9}F_{j+1} + \frac{1}{36}F_{j+2}\right), \qquad (2.16)$$

which is a standard sixth order compact scheme. Using the Taylor expansion, one can prove that if we solve the equation

$\frac{1}{3}\hat{F}'_{j-1} + \hat{F}'_j + \frac{1}{3}\hat{F}'_{j+1} = \frac{1}{\Delta x}(-\frac{1}{36}F_{j-2} - \frac{7}{9}F_{j-1} + \frac{7}{9}F_{j+1} + \frac{1}{36}F_{j+2})$ for $\hat{F}$, then $\hat{F} - F = O(\Delta x^6)$. This was known in [10-12] but it is also a special case of our proof below.

For WCS, we will take the same weight function as in the standard WENO scheme. We will prove that $\omega_i = C_i + O(\Delta x^4)$ (proof is in Section 4) in the smooth region of the solutions but otherwise, $\omega_i = O(1)$. Then using this weight distribution, all WCS implicit candidates become equivalent to

$$\hat{F}_{j+1/2} \approx \omega_{0\,j+1/2}\left(\frac{1}{2}F_{j-1} + \frac{5}{2}F_j + 2\Delta x \bullet F'_j\right)/3 + \omega_{1,j+1/2}\left[\frac{3}{4}(F_{j+1} + F_j) + \frac{1}{4}\Delta x F'_j - \frac{1}{4}\Delta x F'_{j+1}\right]/1.5$$
$$+ \omega_{2,j+1/2}\left(\frac{1}{2}F_{j+2} + \frac{5}{2}F_{j+1} - 2\Delta x \bullet F'_{j+1}\right)/3$$

$$\hat{F}_{j-1/2} \approx \omega_{0,j-1/2}\left(\frac{1}{2}F_{j-2} + \frac{5}{2}F_{j-1} + 2\Delta x \bullet F'_{j-1}\right)/3 + \omega_{1,j-1/2}\left[\frac{3}{4}(F_j + F_{j-1}) + \frac{1}{4}\Delta x F'_{j-1} - \frac{1}{4}\Delta x F'_j\right]/1.5$$
$$+ \omega_{2,j-1/2}\left(\frac{1}{2}F_{j+1} + \frac{5}{2}F_j - 2\Delta x \bullet F'_j\right)/3$$

$$F'_j = \frac{\hat{F}_{j+1/2} - \hat{F}_{j-1/2}}{\Delta x} \approx \frac{1}{\Delta x}\Big[\frac{1}{6}\omega_{2,j+1/2}F_{j+2} + \left(\frac{5}{6}\omega_{2,j+1/2} + \frac{1}{2}\omega_{1,j+1/2} - \frac{1}{6}\omega_{2,j-1/2}\right)F_{j+1}$$
$$+ \left(\frac{1}{2}\omega_{1,j+1/2} + \frac{5}{6}\omega_{0,j+1/2} - \frac{1}{2}\omega_{1,j-1/2} - \frac{5}{6}\omega_{2,j-1/2}\right)F_j$$
$$+ \left(\frac{1}{6}\omega_{0,j+1/2} - \frac{1}{2}\omega_{1,j-1/2} - \frac{5}{6}\omega_{0,j-1/2}\right)F_{j-1} - \frac{1}{6}\omega_{0,j-1/2}F_{j-2}\Big] \qquad (2.17)$$
$$+ \left(-\frac{2}{3}\omega_{2,j+1/2} - \frac{1}{6}\omega_{1,j+1/2}\right)F'_{j+1} + \left(\frac{1}{6}\omega_{1,j+1/2} + \frac{2}{3}\omega_{0,j+1/2} + \frac{1}{6}\omega_{1,j-1/2} + \frac{2}{3}\omega_{2,j-1/2}\right)F'_j$$
$$+ \left(-\frac{1}{6}\omega_{1,j-1/2} - \frac{2}{3}\omega_{0,j-1/2}\right)F'_{j-1}.$$

The weighted compact scheme uses three candidates (Figure 2): $E_0 : F_j, F_{j-1}, F'_j$, $E_1 : F_j, F_{j+1}, F'_j, F'_{j+1}$, and $E_2 : F_{j+2}, F_{j+1}, F'_{j+1}$.

The equation can be written as

$$(\frac{1}{6}\omega_{1,j-1/2}+\frac{2}{3}\omega_{0,j-1/2})F_{j-1}^{'}+[1-(\frac{1}{6}\omega_{1,j+1/2}+\frac{2}{3}\omega_{0,j+1/2}+\frac{1}{6}\omega_{1,j-1/2}+\frac{2}{3}\omega_{2,j-1/2}]F_{j}^{'}+$$
$$(\frac{2}{3}\omega_{2,j+1/2}+\frac{1}{6}\omega_{1,j+1/2})F_{j+1}^{'}\approx\frac{1}{\Delta x}[\frac{1}{6}\omega_{2,j+1/2}F_{j+2}+(\frac{5}{6}\omega_{2,j+1/2}+\frac{1}{2}\omega_{1,j+1/2}-\frac{1}{6}\omega_{2,j-1/2})F_{j+1}$$
$$+(\frac{1}{2}\omega_{1,j+1/2}+\frac{5}{6}\omega_{0,j+1/2}-\frac{1}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{2,j-1/2})F_{j}$$
$$+(\frac{1}{6}\omega_{0,j+1/2}-\frac{1}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{0,j-1/2})F_{j-1}-\frac{1}{6}\omega_{0,j-1/2}F_{j-2}]$$

or

$$\frac{5}{3}(\frac{1}{6}\omega_{1,j-1/2}+\frac{2}{3}\omega_{0,j-1/2})F_{j-1}^{'}+\frac{5}{3}[1-(\frac{1}{6}\omega_{1,j+1/2}+\frac{2}{3}\omega_{0,j+1/2}+\frac{1}{6}\omega_{1,j-1/2}+\frac{2}{3}\omega_{2,j-1/2}]F_{j}^{'}+$$
$$\frac{5}{3}(\frac{2}{3}\omega_{2,j+1/2}+\frac{1}{6}\omega_{1,j+1/2})F_{j+1}^{'}\approx\frac{1}{\Delta x}\frac{5}{3}[\frac{1}{6}\omega_{2,j+1/2}F_{j+2}+(\frac{5}{6}\omega_{2,j+1/2}+\frac{1}{2}\omega_{1,j+1/2}-\frac{1}{6}\omega_{2,j-1/2})F_{j+1}$$
$$+(\frac{1}{2}\omega_{1,j+1/2}+\frac{5}{6}\omega_{0,j+1/2}-\frac{1}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{2,j-1/2})F_{j}$$
$$+(\frac{1}{6}\omega_{0,j+1/2}-\frac{1}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{0,j-1/2})F_{j-1}-\frac{1}{6}\omega_{0,j-1/2}F_{j-2}]$$

Let us solve the equation

$$\frac{5}{3}(\frac{1}{6}\omega_{1,j-1/2}+\frac{2}{3}\omega_{0,j-1/2})\hat{F}_{j-1}^{'}+\frac{5}{3}[1-(\frac{1}{6}\omega_{1,j+1/2}+\frac{2}{3}\omega_{0,j+1/2}+\frac{1}{6}\omega_{1,j-1/2}+\frac{2}{3}\omega_{2,j-1/2}]\hat{F}_{j}^{'}+$$
$$\frac{5}{3}(\frac{2}{3}\omega_{2,j+1/2}+\frac{1}{6}\omega_{1,j+1/2})\hat{F}_{j+1}^{'}\approx\frac{1}{\Delta x}\frac{5}{3}[\frac{1}{6}\omega_{2,j+1/2}F_{j+2}+(\frac{5}{6}\omega_{2,j+1/2}+\frac{1}{2}\omega_{1,j+1/2}-\frac{1}{6}\omega_{2,j-1/2})F_{j+1}$$
$$+(\frac{1}{2}\omega_{1,j+1/2}+\frac{5}{6}\omega_{0,j+1/2}-\frac{1}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{2,j-1/2})F_{j}$$
$$+(\frac{1}{6}\omega_{0,j+1/2}-\frac{1}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{0,j-1/2})F_{j-1}-\frac{1}{6}\omega_{0,j-1/2}F_{j-2}] \qquad (2.18)$$

While the left hand side
=

$$\frac{5}{3}\left\{(\frac{1}{6}\omega_{1,j-1/2}+\frac{2}{3}\omega_{0,j-1/2})\hat{F}'_{j-1}+[1-(\frac{1}{6}\omega_{1,j+1/2}+\frac{2}{3}\omega_{0,j+1/2}+\frac{1}{6}\omega_{1,j-1/2}+\frac{2}{3}\omega_{2,j-1/2})]\hat{F}'_{j}+\right.$$

$$\left.(\frac{2}{3}\omega_{2,j+1/2}+\frac{1}{6}\omega_{1,j+1/2})\hat{F}'_{j+1}\right\}$$

$$=\frac{5}{3}\{[1+\frac{2}{3}\omega_{0,j-1/2}-\frac{2}{3}\omega_{2,j-1/2}-\frac{2}{3}\omega_{0,j+1/2}+\frac{2}{3}\omega_{2,j+1/2}]\hat{F}'_{j}+$$

$$(-\frac{2}{3}\omega_{0,j-1/2}-\frac{1}{6}\omega_{1,j-1/2}+\frac{1}{6}\omega_{1,j+1/2}+\frac{2}{3}\omega_{2,j+1/2})\Delta x\hat{F}''_{j}$$

$$+(\frac{2}{3}\omega_{0,j-1/2}+\frac{1}{6}\omega_{1,j-1/2}+\frac{1}{6}\omega_{1,j+1/2}+\frac{2}{3}\omega_{2,j+1/2})\frac{1}{2}\Delta x^2\hat{F}'''_{j}$$

$$+(-\frac{2}{3}\omega_{0,j-1/2}-\frac{1}{6}\omega_{1,j-1/2}+\frac{1}{6}\omega_{1,j+1/2}+\frac{2}{3}\omega_{2,j+1/2})\frac{1}{6}\Delta x^3\hat{F}^{(4)}_{j}$$

$$+(\frac{2}{3}\omega_{0,j-1/2}+\frac{1}{6}\omega_{1,j-1/2}+\frac{1}{6}\omega_{1,j+1/2}+\frac{2}{3}\omega_{2,j+1/2})\frac{1}{24}\Delta x^4\hat{F}^{(5)}_{j}\},$$

the right hand side =

$$\frac{1}{3\Delta x}[(5\omega_{0,j+1/2}+5\omega_{1,j+1/2}+5\omega_{2,j+1/2}-5\omega_{0,j-1/2}-5\omega_{1,j-1/2}-5\omega_{2,j-1/2})F_j+$$

$$(-\frac{5}{6}\omega_{0,j+1/2}+\frac{5}{2}\omega_{1,j+1/2}+\frac{35}{6}\omega_{2,j+1/2}+\frac{35}{6}\omega_{0,j-1/2}+\frac{5}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{2,j-1/2})\Delta xF_j'+$$

$$(\frac{5}{6}\omega_{0,j+1/2}+\frac{5}{2}\omega_{1,j+1/2}+\frac{15}{2}\omega_{2,j+1/2}-\frac{15}{2}\omega_{0,j-1/2}-\frac{5}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{2,j-1/2})\frac{1}{2}\Delta x^2F_j''+$$

$$(-\frac{5}{6}\omega_{0,j+1/2}+\frac{5}{2}\omega_{1,j+1/2}+\frac{65}{6}\omega_{2,j+1/2}+\frac{65}{6}\omega_{0,j-1/2}+\frac{5}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{2,j-1/2})\frac{1}{6}\Delta x^3F_j'''+$$

$$(\frac{5}{6}\omega_{0,j+1/2}+\frac{5}{2}\omega_{1,j+1/2}+\frac{35}{2}\omega_{2,j+1/2}-\frac{35}{2}\omega_{0,j-1/2}-\frac{5}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{2,j-1/2})\frac{1}{24}\Delta x^4F_j^{(4)}+$$

$$(-\frac{5}{6}\omega_{0,j+1/2}+\frac{5}{2}\omega_{1,j+1/2}+\frac{185}{6}\omega_{2,j+1/2}+\frac{185}{6}\omega_{0,j-1/2}+\frac{5}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{2,j-1/2})\frac{1}{120}\Delta x^5F_j^{(5)}+$$

$$(\frac{5}{6}\omega_{0,j+1/2}+\frac{5}{2}\omega_{1,j+1/2}+\frac{115}{2}\omega_{2,j+1/2}-\frac{115}{2}\omega_{0,j-1/2}-\frac{5}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{2,j-1/2})\frac{1}{720}\Delta x^6F_j^{(6)}+$$

$$(-\frac{5}{6}\omega_{0,j+1/2}+\frac{5}{2}\omega_{1,j+1/2}+\frac{665}{6}\omega_{2,j+1/2}+\frac{665}{6}\omega_{0,j-1/2}+\frac{5}{2}\omega_{1,j-1/2}-\frac{5}{6}\omega_{2,j-1/2})\frac{1}{7!}\Delta x^7F_j^{(7)}$$

Then we assume

$$\omega_0 = C_0 \bullet \frac{9}{5} + O(\Delta x^4) = \tilde{C}_0 + O(\Delta x^4) = \frac{1}{10} + O(\Delta x^4),$$
$$\omega_2 = C_2 \bullet \frac{9}{5} + O(\Delta x^4) = \tilde{C}_2 + O(\Delta x^4) = \frac{1}{10} + O(\Delta x^4),$$
$$\omega_1 = C_1 \bullet \frac{9}{5} + O(\Delta x^4) = \tilde{C}_1 + O(\Delta x^4) = \frac{8}{10} + O(\Delta x^4).$$

Therefore the equation (2.18) becomes:

$$\frac{5}{3}\left\{[1+O(\Delta x^4)]\hat{F}_j' + (O(\Delta x^4))\Delta x \hat{F}_j'' + (\frac{2}{5}+O(\Delta x^4))\frac{1}{2}\Delta x^2 \hat{F}_j''' + (O(\Delta x^4))\frac{1}{6}\Delta x^3 \hat{F}_j^{(4)} + (\frac{2}{5}+O(\Delta x^4))\frac{1}{24}\Delta x^4 \hat{F}_j^{(5)}\right\}$$
$$= \frac{1}{3\Delta x}[(5+O(\Delta x^4))\Delta x F_j' + (0+O(\Delta x^4))\frac{1}{2}\Delta x^2 F_j'' + (6+O(\Delta x^4))\frac{1}{6}\Delta x^3 F_j''' + (0+O(\Delta x^4))\frac{1}{24}\Delta x^4 F_j^{(4)} +$$
$$(10+O(\Delta x^4))\frac{1}{120}\Delta x^5 F_j^{(5)} + (0+O(\Delta x^4))\frac{1}{720}\Delta x^6 F_j^{(6)} + (26+O(\Delta x^4))\frac{1}{7!}\Delta x^7 F_j^{(7)}] \quad (2.19)$$

We denote $\hat{F} - F = R(x)$ the residual term. By substituting into eq. (2.19), we obtain $\hat{F} - F = R(x) = O(\Delta x^6)$ which states that order of accuracy of WCS will also be 6, the same of that in eq. (2.13).

Section 4. The weight functions

We now proceed to analyze the weight functions for WCS.

Proposition 4.1 . *Assume eq (2.1), a 1-D conservation law has a nonlinear function F that is differentiable up to 7th order and $u(x,t)$ is a generalized solution of eq. (2.1). Then for any given fixed bounded domain $D \subset R^1$ and $0 < t \le T < \infty$ there exist $h_0 = h_0(u) > 0$ such that for any mesh size $\Delta x < h_0$, WCS or WENO scheme gives a weight function* $\omega_i = c_i + O(\Delta x^4)$ for regions where $u(x,t)$ *and its derivatives (up to 4th order) are bounded. Otherwise,* $\omega_i = O(1)$.

Proof :

At smooth region of F($u(x,t)$)

$$IS_0 = \frac{13}{12}(F_{j-2} - 2F_{j-1} + F_j)^2 + \frac{1}{4}(F_{j-2} - 4F_{j-1} + 3F_j)^2$$
$$= \frac{13}{12}(F''\Delta x^2)^2 + \frac{1}{4}(2F'\Delta x - \frac{2}{3}F'''\Delta x^3)^2 + O(\Delta x^6)$$
$$= (F'\Delta x)^2 + \frac{13}{12}(F'')^2\Delta x^4 - \frac{2}{3}F'F'''\Delta x^4 + O(\Delta x^6)$$

Similarly,

$$IS_1 = \frac{13}{12}(F_{j-1} - 2F_j + F_{j+1})^2 + \frac{1}{4}(F_{j-1} - F_{j+1})^2$$
$$= \frac{13}{12}(F''\Delta x^2)^2 + \frac{1}{4}(2F'\Delta x + \frac{1}{3}F'''\Delta x^3)^2 + O(\Delta x^6)$$
$$= (F'\Delta x)^2 + \frac{13}{12}(F'')^2\Delta x^4 + \frac{1}{3}F'F'''\Delta x^4 + O(\Delta x^6)$$

$$IS_2 = \frac{13}{12}(F_j - 2F_{j+1} + F_{j+2})^2 + \frac{1}{4}(F_{j+2} - 4F_{j+1} + 3F_j)^2$$
$$= \frac{13}{12}(F''\Delta x^2)^2 + \frac{1}{4}(2F'\Delta x - \frac{2}{3}F'''\Delta x^3)^2 + O(\Delta x^6)$$
$$= (F'\Delta x)^2 + \frac{13}{12}(F'')^2\Delta x^4 - \frac{2}{3}F'F'''\Delta x^4 + O(\Delta x^6)$$

Thus

$$\gamma_0 = \frac{C_0}{(\varepsilon + IS_0)^p} = C_0(\frac{1}{(\varepsilon + (F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{2}{3}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon + (F'\Delta x)^2)^{p+1}})$$
$$\gamma_1 = \frac{C_1}{(\varepsilon + IS_1)^p} = C_1(\frac{1}{(\varepsilon + (F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 + \frac{1}{3}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon + (F'\Delta x)^2)^{p+1}})$$
$$\gamma_2 = \frac{C_2}{(\varepsilon + IS_2)^p} = C_2(\frac{1}{(\varepsilon + (F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{2}{3}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon + (F'\Delta x)^2)^{p+1}})$$

and we got

$$\gamma_0 + \gamma_1 + \gamma_2 = C_0(\frac{1}{(\varepsilon + (F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{2}{3}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon + (F'\Delta x)^2)^{p+1}})$$
$$+ C_1(\frac{1}{(\varepsilon + (F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 + \frac{1}{3}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon + (F'\Delta x)^2)^{p+1}})$$
$$+ C_2(\frac{1}{(\varepsilon + (F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{2}{3}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon + (F'h)^2)^{p+1}})$$
$$= (\frac{1}{(\varepsilon + (F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{1}{15}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon + (F'\Delta x)^2)^{p+1}})$$

because of the fact $C_0 + C_1 + C_2 = 1$.

Finally we derive

$$\omega_0 = \frac{\gamma_0}{\sum_{i=0}^{2}\gamma_i} = \frac{C_0\left(\frac{1}{(\varepsilon+(F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{2}{3}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}\right)}{\left(\frac{1}{(\varepsilon+(F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{1}{15}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}\right)}$$

$$= C_0\left(1 + \frac{\left(p\frac{\frac{3}{5}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}\right)}{\frac{1}{(\varepsilon+(F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{1}{15}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}}\right),$$

and

$$\omega_1 = \frac{\gamma_1}{\sum_{i=0}^{2}\gamma_i} = \frac{C_1\left(\frac{1}{(\varepsilon+(F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 + \frac{1}{3}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}\right)}{\left(\frac{1}{(\varepsilon+(F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{1}{15}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}\right)}$$

$$= C_1\left(1 + \frac{\left(-p\frac{\frac{2}{5}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}\right)}{\frac{1}{(\varepsilon+(F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{1}{15}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}}\right),$$

$$\omega_2 = \frac{\gamma_2}{\sum_{i=0}^{2}\gamma_i} = \frac{C_2\left(\frac{1}{(\varepsilon+(F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{2}{3}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}\right)}{\left(\frac{1}{(\varepsilon+(F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{1}{15}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}\right)}$$

$$= C_2\left(1 + \frac{p\frac{\frac{3}{5}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}}{\frac{1}{(\varepsilon+(F'\Delta x)^2)^p} - p\frac{\frac{13}{12}(F'')^2\Delta x^4 - \frac{1}{15}F'F'''\Delta x^4 + O(\Delta x^6)}{(\varepsilon+(F'\Delta x)^2)^{p+1}}}\right).$$

If $\varepsilon > 0$ is a fixed positive constant, and F is smooth in the sense that all its derivatives are bounded, then the weights $\omega_i = C_i + O(\Delta x^4)$. This is in fact a stronger result than the result of [9] that $\omega_i = C_i + O(\Delta x^2)$.

5. Numerical results

We first test WCS scheme on a 1-D linear wave equation with discontinuous initial function:

$$u_t + u_x = 0, \qquad u(x,0) = \begin{cases} 1.0 & if \quad 0.1 \le x \le 0.4 \\ 0.5 & otherwise \end{cases}. \qquad (3.1)$$

The calculation stops at $t = 0.3$ and the solutions are illustrated in figure 3. The results indicate that standard compact scheme is not suitable for shocks while both WCS scheme (labeled UWCNC) and WENO scheme (Labeled WENO-5) work. Furthermore, WCS has less dissipation than WENO near shocks which means a sharper transition is obtained.

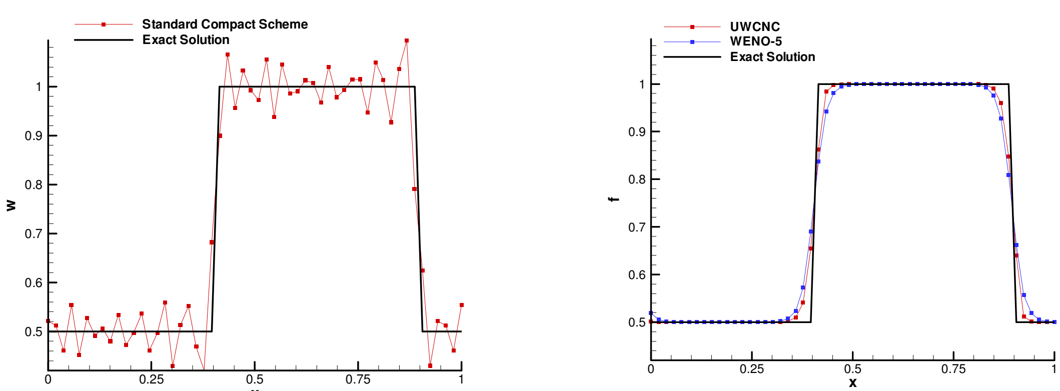


Figure 1. Numerical test over linear wave equation.

To further test the capability of the new scheme in both shock capturing and resolution, we applied it to the 1-D problem of shock/entropy wave interaction. In this case, 1D Euler equations:

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}=0$$

(3.2)

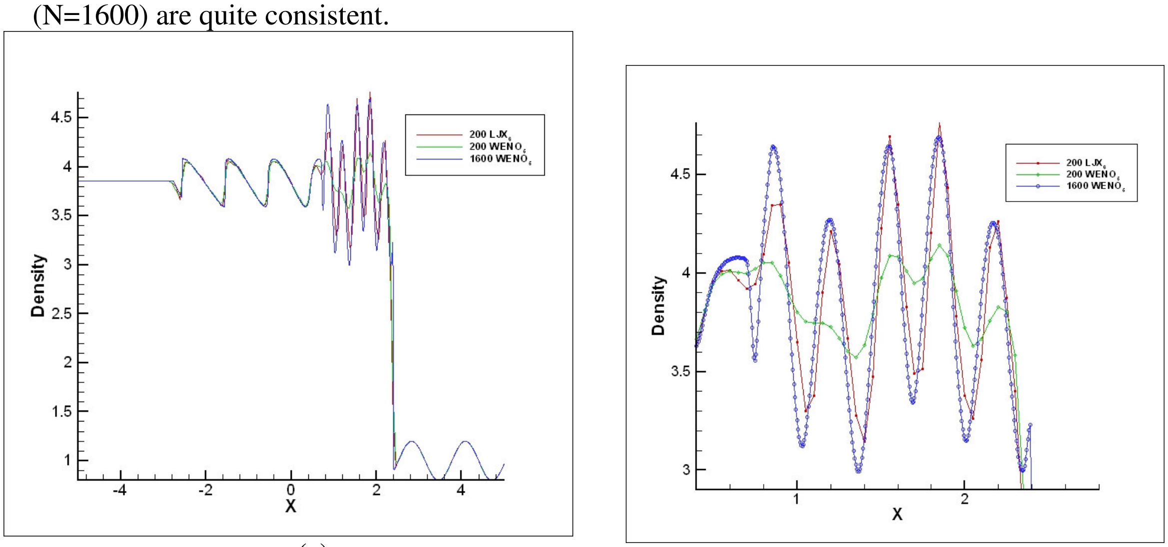
$$U=(\rho,\ \rho u,\ E)^T\ ;\ \ F=(\rho,\ \rho u+p,\ u(E+p))^T$$

are solved with the following initial conditions:

$$(\rho,\ u,\ p)_0=\begin{cases}(3.857143,\ 2.629369,\ 10.33333), & x<-4;\\ (1+0.2\sin(5x),\ 0,\ 1) & x\geq -4.\end{cases}$$

(3.3)

Figure 2 (a) and (b) depict the solutions of the density distribution respectively. On the coarser grid with grid number of N=200, our new scheme (labeled 200 LJK) shows much better resolution for small length scales than the 5th order WENO (labeled 200 WENO). Apparently, there is an order difference in resolution between our 6th order WCS scheme and the 5th order scheme. The numerical results by our WCS scheme with 200 grid points are even comparable with the 5th order WENO scheme (labeled 200 LJK) with 1600 grid points (labeled 1600 WENO) (Figures 2 (a) & (b)). In addition, the WCS captures the shock in a much sharper way for all shocks. On the shocks developed by the sinuous waves, only one grid point was found on the shock (Figure 2 (a)). Again, Figure 3 shows the smoothness measured defined to be a combination of IS_0, IS_1 and IS_2 to detect the shock. Figure 3 shows the main shock is well captured with smoothness $\alpha=1$ (where smooth points are typically have smoothness $\alpha=0$) and the shocks developed by the sine function are also well captured. The smoothness measured on the coarser grid (N=200 and 400) and finer grid (N=1600) are quite consistent.



(a) (b)

Figure 2 Numerical test for 1D shock-entropy wave interaction problem, t=1.8, N=200

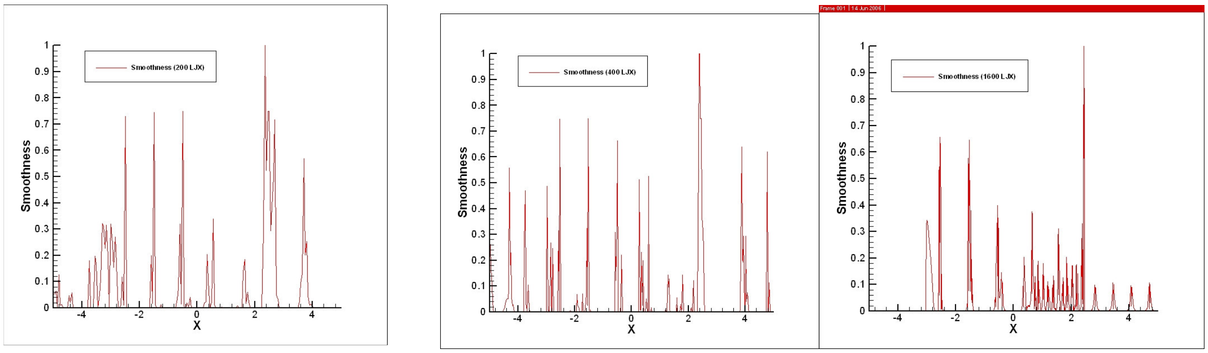


Figure 3 Smoothness for 1D shock-entropy problem, t=2, N=200, 400, 1600

In summary, we have shown both WENO and WCS are good schemes in dealing with shock but they have different strength. Currently we are developing a way to take advantages of both schemes by either averaging both or using one as a filter of another.

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