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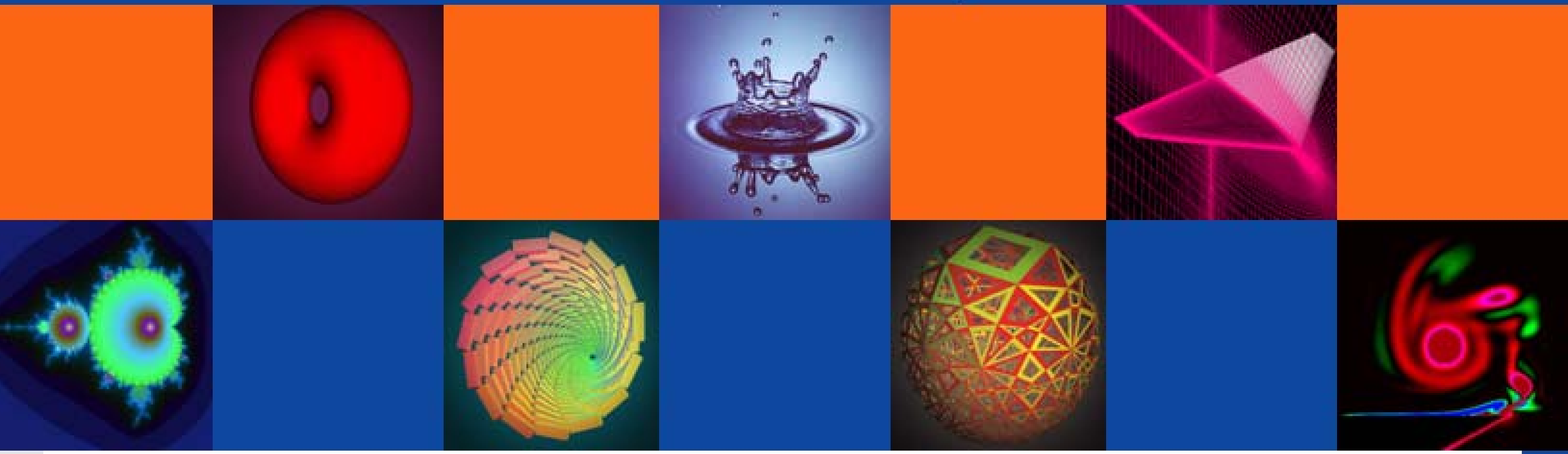

Engaging the Honors Student in Mathematics

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1. Introduction

1.1 Mathematics in the Honors Curriculum

In 2005, the National Collegiate Honors Council developed and published a set of twelve characteristics of a fully developed honors college. The two that refer to curriculum are 1) the curriculum of a fully developed honors college should offer significant course opportunities across all four years of study, and 2) the curriculum of a fully developed honors college should constitute at least 20\% of a student's degree program; an honors thesis or project should be required.

To satisfy these requirements, institutions offer the following honors opportunities: honors courses for general education requirements, honors courses in the major or minor discipline, special topic upper division honors seminars, special topic lower division honors seminars, special topic interdisciplinary honors seminars, honors senior thesis/creative project, honors independent study, undergraduate research courses, honors study abroad opportunities, honors internships, and service learning courses.

Peterson's *Smart Choices: Honors Programs and Colleges* (2005) [10] details admission standards, degree requirements, and general program descriptions for the 84 honors colleges in United States institutions of higher learning. The average number of honors credits required is 25.1 credits, with a range from 10 to 68.

When it comes to satisfying course requirements in mathematics, the most common traditional courses offered are Honors Calculus I, Honors Calculus II, and honors mathematics for non-mathematics majors, for example, Liberal Arts Honors Mathematics at the University of Texas at Arlington (UT Arlington). Honors students majoring in mathematics and those interested in higher mathematics satisfy the honors credit requirements by "contracting" several of their upper division mathematics classes.

Because of our experience in teaching honors courses at UT Arlington, our colleagues approach us with questions such as the following. How do I make a calculus course "honors"? How should the mathematics in a course for non-majors be taught so that it provides an "honors" experience for the students in the class? What are suggestions

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for designing an honors contract for an upper division mathematics class? What are some ideas for projects for mathematics honors theses? This article will address these and other questions regarding mathematics in the honors curriculum.

1.2 Characteristics of an Honors Student

Who is an honors student? Answering this question is a starting point for discussing how to engage the honors student in mathematics. Drawing on our years of teaching honors courses, we have observed the following characteristics of honors students:

- Honors students have the desire and motivation during their undergraduate years to educate themselves beyond the requirements of the degree they seek.

- Honors students want to understand what they study in greater depth and within a broader context, with a vision toward development of a career and lifelong learning.

- Honors students are actively engaged in their learning, taking ownership of their education.

- An honors student is in a class because he/she wants to be there.

Because of their intrinsic desire to "get more" out of their classes, teaching honors students opens up new possibilities and challenges for the teacher.

1.3 Philosophy of Teaching Honors Students in a Mathematics Course

Here we outline the philosophy that forms the foundation for the more specialized discussions that follow in Sections 2 through 5.

1. Honors students are the owners of the mathematics that they study. The honors student must take the initiative in deciding whether mathematical statements are true or false, whether a question is worthy of investigation or not, and how new mathematical concepts should be formally defined.

2. In an honors course, students should communicate and defend their arguments, both formally and informally, both orally and in writing, to the instructor and to their classmates.

3. An honors course should expand the students' view of what mathematics is and how to think about it so that after they complete the course students can look back and be amazed at how their mathematical maturity has developed.

4. An honors course in mathematics should give the student a perspective on how the subject has developed and how the subject is still evolving. This can include how other disciplines have influenced the development of mathematics or how mathematics has driven advances in other disciplines or among other fields of mathematics.

2. Mathematics for the Honors Liberal Arts Student

Many students, even honors students, enter mathematics courses with a fear of mathematics. At UT Arlington we teach an honors mathematics course designed for honors liberal arts students. In this course, fear of mathematics is the invisible gorilla in the room at the beginning of the semester. The most represented majors in this class tend to be English and Journalism, and as a whole, the students do not have much confidence in their ability to do mathematics. However, since the course is designed so that the students can discover and explore topics in mathematics that they (and even undergraduate mathematics majors) may have never heard of, they seem to leave behind many of their mathematical hang-ups and open their minds to the possibility of enjoying mathematics. The mathematical situations which we study are often simple to state, but incredibly rich in their depth. The students encounter and interact with mathematical areas that have open, yet understandable questions. They are empowered by this knowledge. They are expected to do mathematics that they initially believe to be far beyond their abilities, and it is amazing how they rise to the occasion. Students in the classes that we have taught recently have made the following comments on their course evaluations: "This course made math fun, in spite of my being a Liberal Arts major!" "Math was exciting and something new. I learned how to think outside of the box. It was very different from anything I have ever taken." "It was a great class - especially the less traditional areas of mathematic ‘intrigue’."

Honors mathematics for Liberal Arts majors offers the opportunity to study all sorts of mathematics which are accessible to students at the college freshman level, but which have been omitted from the high school mathematics curriculum because they do not fit in any obvious way with the already crowded traditional sequence of Algebra, Geometry, and Trigonometry. Textbooks that we have used as a source for topics, discovery problems, and projects include *The Heart of Mathematics, An Invitation to Effective Thinking* [1], *To Infinity and Beyond* [9], and *Knots and Surfaces: A Guide to Discovering Mathematics* [5]. We also supplement with personal favorites, articles from journals such as the Notices of the AMS and the American Mathematical Monthly, and occasionally an interesting movie on mathematics. Some of the topics we have included in the past are graph theory, knot theory, the mathematics of voting, fair division, cryptography and coding theory, Fibonacci numbers, the Golden Rectangle, and notions of infinity.

The course also includes a few topics that the students have either studied or heard of, but treats them in a new way. For example, the students invariably recall having studied the Pythagorean Theorem, and many of them are able to state it and use it correctly. The challenge comes when the students are asked, "How do you know that the Pythagorean Theorem is true?" Maybe there is a right triangle for which it does not work. "And, how do you know that it only works for right triangles and not for any other kind of triangle?" For many of the students, this may be the first time they have considered the question of "why" in mathematics. Now, not only are they confronting this question, but they are asked to discover a solution themselves and defend their answers! Suddenly the class is brought to the level of an honors course. The instructor is challenging the students to question a mathematical statement, to discover a solution, and to communicate and defend their answers. Usually at least two class periods are spent on this project. The students gather in groups of three to four around tables and work with cut-outs to devise a geometric proof of the Pythagorean Theorem. The book The *Heart of Mathematics, An Invitation to Effective Thinking* [1], provides the cut-outs in the kit that comes with the textbook. In this project, there are four identical right triangles and one square. These five shapes can be arranged in two ways: (1) as a large square with the hypotenuses of the four triangles as the edges and the cut-out square in the center, and (2) as two concatenated squares, as shown below in Figure 1.

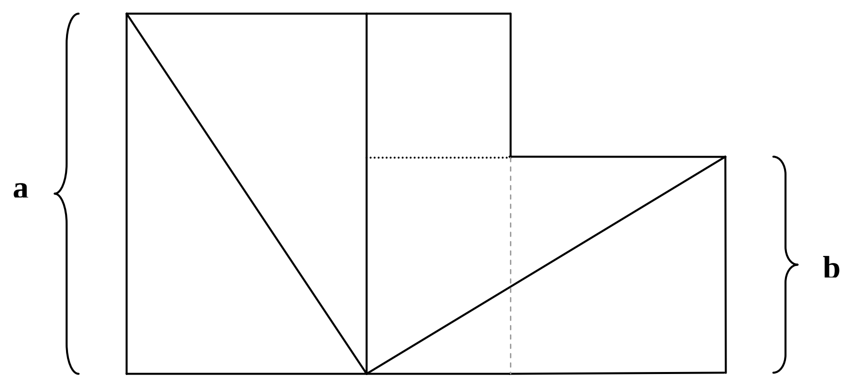

Figure 1

After the students have found both configurations, they are then asked to calculate the area of each configuration by finding the area of the large square in the first case (which is the square of the hypotenuse of the right triangle) and by adding the areas of the two smaller squares in the second case (which is the sum of the squares of the legs of the right triangle, $a^2 + b^2$). Equating these areas then yields the Pythagorean Theorem.

The instructor's role in this exercise is to give guidance without suggesting answers. Questions from the students are met with more questions from the instructor, which should lead the students to discover the solution. After some time, the students discover how, when the pieces are aptly arranged, the Pythagorean Theorem seems to appear. They are then asked to explain their discoveries verbally to the class and, with guidance from the instructor, an algebraic explanation of their picture emerge. In this way, the students become the owners of the theorem; they can now believe that what they were taught is true, not because someone told them it is true, but because they discovered and proved it themselves.

Part of the assignment associated with the project is to have the students write up their proofs, either individually or in small groups. There is opportunity here to help the students (many of whom are English majors!) to write precisely. What they write should say what they mean, but often it does not. An instructive exercise is to quote anonymously exactly what is written in students' homework for the class to consider. In most cases, the student who wrote the statement will look in amazement and completely agree that the statement needs to be changed. It is essential in an honors course to make it clear that papers are expected to be neatly written and complete in the explanations, as if writing to someone who does not already know the solution. For honors students, communication is not a problem, but communicating mathematically is a new twist. One way in which students' mathematical confidence grows is in the realization that a mathematical argument does not need to consist of a two column proof, but a convincingly rigorous argument. These students love to discuss ideas, and so it becomes natural for them to build their understanding of the mathematics by verbalizing it. It is wonderful to hear students who considered themselves math-phobic at the beginning of the semester, heatedly and reasonably arguing about mathematics, for example, about whether you can take something away from a set and still have a set of the same cardinality.

In discussions like this, we let the students decide whether comparing the sizes of infinite sets is an interesting question or whether it even makes sense to talk about it. In their groups, students are asked to consider questions such as the following. Are some infinite sets larger than others? If we take something away from an infinite set, must the set become smaller? What do "larger" and "smaller" mean when we are talking about infinite sets? The class is often split on their views of infinity, and the students are asked to defend their positions verbally to convince those who may think otherwise. Examples and counterexamples are extremely helpful in the class discussions. For instance, a student may argue that there are more counting numbers than even counting numbers (denoted $\mathbf{N}$ and $2\mathbf{N}$, respectively) by showing that the first set properly contains the second. Now the instructor can throw in a twist by demonstrating a one-to-one correspondence between the two sets. The students must now defend or modify their views in light of this.

If the students still seem to believe that $\mathbf{N}$ contains more elements than $2\mathbf{N}$, despite the perfect pairing between them, the instructor can then throw in a further twist. Imagine that we have two bags of dice. The first contains red dice, numbered from 1 to infinity on one face. On the opposite face of the die numbered $\mathbf{n}$, the number $\mathbf{2n}$ is written. The second bag contains blue dice, numbered exactly as the red dice. Now imagine lining up the red dice in increasing order, with the counting numbers showing upward from 1 to infinity. Under each red die place a blue die so that the blue die with the number $\mathbf{2n}$ facing upward appears just under the red die with the number $\mathbf{n}$ facing upward. Now the red dice and the blue dice are paired in a one-to-one correspondence. The students are asked again if they still believe, despite this pairing, that there are more red dice than blue dice because of the numbers showing on the faces. If they are inclined to think that there are still more red dice, then the instructor simply turns all the red dice and all the blue dice upside-down (in this imaginative experiment) so that the even counting numbers now appear on the red dice and the counting numbers on the blue dice.

Did the change in labels change how many red and blue dice there are? What seems to be the reasonable solution that will be consistent with the facts now presented?

Through these types of discussions, the students are led to formulate, with the instructor's guidance, an appropriate definition of what it means for two infinite sets to have the same size. They will recognize that they have no alternative but to accept the often counterintuitive outcomes: One can take an "infinite pizza" and split it into two "pizzas", each exactly as large as the original. There are exactly as many even integers as integers. And, there are exactly as many points between 0 and 1 as there are on the whole infinite real line! Someone encountering such statements for the first time may think that mathematics is simply outlandish, but our honors students, from any major, can now explain that this makes perfect sense; this is the only way it could be, after all, and anything other than that would be preposterous to their now refined intuition.

The essays "Cantor's new look at the infinite" and "To infinity and beyond" from the collection of essays in *To Infinity and Beyond* [9] offer a historical perspective on the development of the mathematical concept of cardinality. Students are often relieved to learn that the same questions they had been considering gave the most brilliant mathematicians at least as hard a time and caused at least as many arguments among them. Even Kronecker, Georg Cantor's mentor, refused to accept Cantor's rigorous and ingenuous formulation of cardinality.

Classes are typically a mixture of lecture mingled with discussions among the students in small groups. Throughout the semester, students report to the class the results of their group discussions and report on areas of mathematics in which they have an interest. This culminates in formal small group presentations to the class at the end of the semester. Some of the recent topics on which students have become "the house experts" include chaos, interconnections between mathematics and music, and the role of game theory in jury selection and medical decision-making.

3. Honors Calculus

An Honors Calculus sequence of courses is a wonderful opportunity to build a mathematical learning community among students. We offer a year-long sequence, Calculus I and Calculus II, as honors courses. Roughly the same group of students take both semesters of the course, so they get an academic year of exposure to the same instructor and the same peers. As well, many of the students, who are primarily mathematics and science majors, share common schedules in their other courses. So the learning community that gets fostered in their Calculus course spills over and supports interactions in their other courses.

In a traditional calculus class, much of the time is spent understanding and practicing techniques for the computation of limits, derivatives, and integrals. As all mathematics instructors know, it's quite easy for the students to lose the forest for the trees. In an honors course, one has the luxury of requiring and expecting the students to pull back and understand how the topics fit into their cumulative mathematical knowledge.

Different instructors have chosen different texts from which to build the course. One instructor chooses a reform calculus text [7], while another uses the standard text

[11] used by the non-honors calculus classes. However, it is the structure of the course rather than the textbook that provides the enrichment expected in an Honors class.

Each week of Honors Calculus consists of three hours of lecture, and two hours of lab. Instead of conducting recitation in the traditional way, by simply going over homework problems, each lab session is structured around a worksheet of problems, written by the instructor, that are much more challenging and far-reaching than those assigned from the textbook. (For some examples of lab worksheets, see the appendix.) The students work in groups to find effective methods of problem-solving. This serves many purposes. First, the students develop working relationships with the other students in class, relationships that hopefully will carry through to other courses that they will take together. Second, the students develop a sense of mathematical confidence. That is, since they become accustomed to the instructor answering all of their questions with more questions, they start to develop an ability to know when they are on the right track, and answer their own leading questions. This lab structure is modeled in part after the format utilized by the Emerging Scholars Program at the University of Texas at Austin and the MathExcel Program at the University of Nebraska-Lincoln. For a complete set of worksheets see [2] and [3] in the references.

The problems that the students work on in lab sessions are varied. There are two types of problems that we have used over and over. The first type is the problems requiring a "flip side" of understanding. For these problems, the students are put into the role of the teacher in that they need to create problems or find examples fitting given specifications to illustrate key ideas. An example of a problem of this type would be the following.

Find two functions $f(x)$ *and* $g(x)$ *and a constant c such that* $\lim_{x\to c} f(x)$ *does not exist,* $\lim_{x\to c} g(x)$ *does not exist, but* $\lim_{x\to c}(f(x)+g(x))$ *does exist.*

To solve this problem is not enough to just know how to compute a limit. The students must put their knowledge of functions into the context of limits to create a counterexample to a common "mistaken theorem" that calculus students often try to apply. First they must think about how to build a function for which the limit does not exist at some value $x = c$. Then they need to consider how they can "fix" the "bad behavior" by adding another function. There are infinitely many possible solutions to this problem, but a simple solution would be to let $f(x) = \frac{1}{x}$ and $g(x) = -\frac{1}{x}$ and consider their behavior at $x = 0$.

Here is an example of another problem from this category.

For each of the following situations, find an appropriate function which satisfies the given requirements. The limits below can be taken to be one-sided if it is convenient for your example.

a) $f(\pi)$ *has the form* $0*\infty$ *and* $\lim_{x\to\pi} f(x) = \infty$.

b) $g(\pi)$ *has the form* $0*\infty$ *and* $\lim_{x\to\pi} g(x)=0.$

c) $h(\pi)$ *has the form* $0*\infty$ *and* $\lim_{x\to\pi} h(x)=5.$

This problem requires students to not only know how to apply l'Hôpital's Rule, but to really understand what an indeterminant form means, and again, draw on their function knowledge to create an example. Moreover, they need to explicitly exhibit the indeterminant nature of the form $0*\infty$ by constructing examples of functions with a prescribed limit. This is not an easy problem for students, even honors students, but after struggling through the ideas, they have a better understanding of the concept of indeterminant forms. Again, there are many solutions to this problem. Here are some possibilities.

a) $f(x)=(x-\pi)e^{\frac{1}{x-\pi}}$

b) $g(x)=-(x-\pi)\ln(x-\pi)$

c) $h(x)=5\sin x\tan(x-\frac{\pi}{2})$

The second problem type which is emphasized on the lab worksheets are those which inevitably begin with the phrase "explain in your own words." These may be interpretation problems, such as explaining what a derivative means for a concrete example. For instance,

A company's revenue from computer sales, R, measured in thousands of dollars, is a function of advertising expenditure, a, also measured in thousands of dollars. Suppose $R=f(a)$. *Explain what the statement* $f'(301)=2$ *means in practical terms.*

It is important, especially on these interpretation problems, to require the students to really answer the question. Students will often times solve the previous problem by saying, $f'(301)=2$ means the derivative of *f* at 301 is 2." It becomes the instructor's job to draw out the answer by asking scaffolding questions to guide the students to fully consider the problem. Here are some examples of scaffolding questions for this problem.

What are the units of the number 301?
What are the units of the number 2?
What does *f* measure?
Do you have any information about the value of $f(301)$? Do you need it?
How would the computer company use information about *f'*?
If the company is already spending $301,000 on advertising, would it be wise for the company to increase its advertising expenditures?
If $f'(301)=0.3$? Would your answer to the previous question change?

A complete answer to this problem should be something like

> $f'(301) = 2$ *means that if the company is already spending $301,000 on advertising and it spends a little bit more on advertising, it would expect its revenue to increase by approximately twice the amount of increase in advertising expenditure. For instance, if it spends $301,100 on advertising, it would expect its revenue to go up by about $200, so it would make back the extra $100 it spent on advertising, plus $100 more.*
>
> Or the problems may be of a more conceptual type, such as:
>
> *Give an explanation of the Mean Value Theorem that a precalculus student could understand. Your explanation should be both verbal and pictorial.*

These problems seek to make connections between the computational and conceptual ideas of calculus and hone the honors students' communication and justification abilities. For some sources of good problems, see references [7] and [4] (particularly "Using geometric models to predict convergence" on pages 95-98).

Students are expected to periodically present their problem solutions and ideas to the rest of the class, further reinforcing the need to communicate mathematics. To encourage the students to keep up with the regular homework assignments, one lab each week starts with ten minutes of presentations of homework problems by students. Three homework problems assigned the previous week are selected by the instructor. The students are not informed ahead of time which problems will be presented. Three students are randomly selected to present homework problems. Each student is given the option to present. If the student is not at lab that day, he gets zero points for his presentation. If the student is at lab, but does not feel ready or willing to present the requested problem, he gets two points for attendance. If the student presents a solution, five points are given. The atmosphere during the presentations is not stressful or high pressure. It is meant to be an opportunity for the students to get practice at communicating mathematics and refining their solutions. Again, it is important to emphasize to the students that mathematics needs to be communicated, and it is only by offering one's work for criticism from one's peers that mistakes can be found. This is true of the way professional mathematicians do mathematics, and can be true of an undergraduate mathematics classroom as well.

The students also work collaboratively on extensive group projects assigned outside of class time that require not only problem-solving, but good communication, in the form of a written report. The students are given a month and a half to complete the projects, and the final product is expected to be of high mathematical quality and well-written in all respects. A recent semester group project was based on designing a suspension bridge to satisfy prescribed dimensions. The students needed to determine the length of a catenary supporting the bridge, using only their Calculus I knowledge. They hadn't yet encountered the arc length formula, and through the project, the students developed the formula. Another Calculus I project involved measuring the volume of wine in a barrel with a bung rod after finding the optimal barrel dimensions (see [12]). A recent Calculus II project had the groups finding the generating function of the Fibonacci sequence via Taylor series. These group projects reinforce the cooperative atmosphere of the classroom, requiring the students to work together extensively on their own time.

As anecdotal evidence of the successful creation of an atmosphere of enjoyable community learning, when the instructor of the 2006 Fall semester Honors Calculus class arrived at her classroom to administer the final exam, she found a room decorated with streamers and confetti, a buffet of home-baked cupcakes, and students blowing bubbles, lying in wait to celebrate their final exam. The students truly feel like the class is their own, and their peers are their collaborators. (Though not on the final exam.)

4. Honors Contracts in Mathematics Courses

An honors program may allow students to take a non-honors course for honors credit through an honors contract between the student and the instructor, in which honors requirements are laid out for the student to complete during the course. Since the course itself is not taught as an honors course in the sense that the majority of the students taking it are not honors students, the instructor must carefully craft the honors component to provide a truly honors experience for the one or few students who are taking it for honors credit. We have used several approaches in designing the material required for the contract courses. One approach is to require the honors students to write a paper that delves more deeply into a topic in the class that has sparked their interest. Here the student can investigate how the mathematical ideas developed and how they are connected with other branches and over-arching ideas in mathematics. For example, an honors student in a recent section of Real Analysis I at UT Arlington wrote a paper on the formal development of the real number system, starting with the formal definition of an integer, then leading to the definition of a rational number, and then to Dedekind cuts, which yields the real number system. A paper of this kind should be started early in the semester so that the student and instructor can discuss the progress and further directions of the paper. It is important to keep the honors student on a schedule with regard to writing the paper, since even honors students are pressured by class work to procrastinate on serious projects if not guided properly! The honors student can be asked to prepare a synopsis of the paper to present to the class or to the department's mathematics club near the end of the semester. The topic for such a project should ideally come from the student. The paper can also mention or discuss problems in the subject that are not yet solved.

Another type of honors project that expands the students' view of the mathematical world is to expose them to the mathematics literature. Journals that contain articles on mathematics that are suitable for honors mathematics majors to tackle include the *American Mathematical Monthly, the Mathematics Magazine, the College Mathematics Journal* (all publications of the Mathematical Association of America), and the *Notices* or the *Bulletin* of the American Mathematical Society. The honors student may be asked to find an article in one of these journals that is related to the course and write a synopsis of it explaining what its purpose is and how the mathematics ties in with and expands upon what was done in class. Examples of such articles include "The historical development of infinitesimal mathematics" [8] (appropriate for a course in calculus or analysis), "Pythagorean triples and inner products" [6] (suited for a course in modern algebra or linear algebra), and "Lifting the curtain: Using topology to probe the hidden action of enzymes" [12].

Yet a third approach is to have the honor students each week work on some additional problems-usually the harder more in-depth problems in the textbook, or from a more advanced book. Actually these selected additional problems could be announced to the class to motivate all students to work on these challenging questions. The honor students will present their solutions orally each week to the instructor. The instructor may also select some of these problems for the students to present to the class two or three times during the term. This idea, once again, reinforces the importance of communication in mathematics and how, by explaining one's solutions, one gains a better understanding of the ideas used in that solution. Using this approach ensures that the honors student is gaining a deeper understanding of the material in the course by acquiring knowledge through their own investigations. It also provides consistency throughout the semester.

Finally, the fact that the course is not itself an honors course can be used positively. In almost every non-honors mathematics course, there are students who are struggling to learn the material and keep up with the homework. Here the honors student has a unique opportunity to communicate mathematics by leading several working sessions during the semester, possibly before exams in the course. It is both challenging and enlightening mathematically for the honors student to think about how to explain mathematics to others. In consultation with the instructor, the honors student is asked to prepare sessions in which he guides the students, as they work with each other in solving selected problems. At UT Arlington, a contract honors student in Analysis I conducted a study session each week in Analysis I the following semester, sponsored by the UT Arlington Chapter of the MAA. Both the students and the honors mentor matured mathematically from this experience and enjoyed working together. Providing such an experience for the honors student attends to the teaching principles of expanding the honors student's way of thinking about mathematics and having the honors student communicates mathematics. It also satisfies the principle that honors students should take ownership of the mathematics since it is their task to determine how they will explain it to their peers and interact with them as their mentor.

5. The Senior Honors Thesis Project

Many honors programs require the student to write an honors paper or thesis, usually during the final two semesters of the degree. The honors paper typically involves some form of research, which may be an analysis of existing research literature to support a thesis statement of the student. In an honors project, the creativity, the new ideas, and the motivation come from the student. The mentor should allow the student's questions, interests, and enthusiasm to lead the development of the paper, providing guidance in the organization, in finding references, and in understanding the mathematics. The project should culminate in a formal presentation of the paper by the student to an audience of both faculty and students, with time for questions afterwards. Such a presentation gives the honors students a sense of having "defended" an original piece of work as part of their degree, in preparation for perhaps further studies in graduate school including the defense of a dissertation. At UT Arlington, for example, every graduating honors students is required to give a fifteen-minute presentation before faculty judges at the *Honors Undergraduate Research and Creative Activity Symposium*, or the *Academic Celebration of Excellence by Students Symposium*, with awards given for the presentations judged to be the best.

During the first semester of work on the honors thesis project, students formulate their proposal, seeking out and studying research literature to guide its development. This is perhaps the most difficult part of the work in that the direction of the project often changes its course as the students encounter ideas that may challenge their previous understanding. It is important that the mentor not force a direction that the student may have initially had in mind since this can stifle their creativity and cause them to lose interest in the project. It is helpful during this stage for the student to talk to other faculty, in the department of mathematics or outside the department, if their paper involves other disciplines. Communicating with others is one of the best ways for the students to encounter issues related to their work that they may not have considered before. The best strides during this phase of the work occur when students have an insight of their own and excitedly want to tell others about what they have discovered. Here is where the paper begins to take its shape and becomes the true work of an honors student.

By the beginning of the second semester, students will have formulated a proposal, studied some new mathematics, and collected the research literature they need to construct their paper. Now the mentor's role is to guide the organization of the paper and its formal structure in terms of citations and style. Motivation to keep a good writing schedule is afforded by scheduling the formal presentation of the paper early in the semester so that the student has a date to look forward to. Before the presentation, the mentor should spend time helping the student with good presentation techniques such as considering what the audience may or may not already know, not including too much material to cover in the allotted time, making the slides easy for the audience to see, and talking directly to the audience, rather than reading a prepared speech. The mentor should also discuss how to handle questions from the audience, including those to which the presenter may not have an immediate answer. All these things are important and necessary in developing a new generation of mathematicians and mathematics teachers who can think creatively, write well, and communicate effectively.

A recent paper by an honors mathematics student at UT Arlington deals with the role of Lie theory in the development of the standard model of particle physics; another studied how to improve the teaching of high school algebra by motivating each new idea through practical discussions relevant to the students. In each case, the student either initially proposed the idea or took a proposal by the mentor and developed it in their own way. Each of these papers had the unique signature of an honors student, offering to the reader the carefully organized and original work of a creative and budding mathematical mind.

6. Conclusion

While incorporating mathematics in the honors curriculum comes in many different guises, the common thread is high expectations, for both the students and the instructor. The instructor has the opportunity to utilize more options for student interaction and assessment, ones which incorporate high standards of communication, encourage student ownership of the mathematics being learned, and give students a real feel for the profession of mathematics. Though the ultimate goal is to create an enriched learning situation for the students, the successful engagement of honors students in mathematics is hugely rewarding for the instructor, and is an experience we hope this monograph facilitates.

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Appendix

Sample Honors Calculus I Worksheet

1. For each situation, sketch the graph of a function f that satisfies the given condition:
   (a) $\lim_{x\to 5} f(x)$ exists, however f is not continuous at $x = 5$.
   (b) f is continuous on $(-\infty, 1)$ and on $(1, \infty)$, but f is not continuous on $(-\infty, \infty)$.
   (c) f has domain $[0, 5]$ and is continuous on $[0, 5)$, but is not continuous on $[0, 5]$.
   (d) f is continuous everywhere except at $x = 5$, at which point it is continuous from the right.
   (e) f is discontinuous at $x = 5$, and $f(5) = 1$, however, this function f is such that if $f(5) = 1$ were changed to $f(5) = 0$, then f would be continuous at $x = 5$.

2. Draw the graph of an interesting, non-symmetric function $f(x)$ which is differentiable on $(-2, 4)$. Without doing any computations, that is, purely geometrically, order each of the following from least to greatest for your function.
   $$f'(0), \quad \frac{f(2) - f(0)}{2}, \quad f'(3), \quad f'(-1), \quad \frac{f(0) - f(-1)}{1}$$

3. Define a function f that has domain $\mathbf{R}$, but is continuous NOWHERE.

4. For each of the following, you must completely justify your answer, with either a convincing argument or a counterexample.
   (a) True or False. As x increases to 100, $f(x) = 1/x$ gets closer and closer to 0, so the limit as x goes to 100 of $f(x)$ is 0.
   (b) True or False. $\lim_{x\to a} f(x) = L$ means that if x_1 is closer to a than x_2 is, then $f(x_1)$ will be closer to L than $f(x_2)$ is.
   (c) You are trying to guess $\lim_{x\to 0} f(x)$. You plug in $x = 0.1, 0.01, 0.001, \ldots$ and get $f(x) = 0$ for all these values. In fact, you're told that for all $n \in \mathbf{N}, f(\frac{1}{10^n}) = 0$. True or False. Since the sequence 0.1, 0.01, 0.001,... goes to 0, we know $\lim_{x\to 0} f(x) = 0$.
   (d) If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)}$
      i. does not exist
      ii. must exist
      iii. not enough information
   (e) The statement "Whether or not $\lim_{x\to a} f(x)$ exists depends on how $f(a)$ is defined," is true
      i. sometimes
      ii. always
      iii. never
   (f) Suppose you have two linear functions f and g. The points $(0, 6)$ and $(a, 0)$ lie on the graph of f, and the points $(0, 3)$ and $(a, 0)$ lie on the graph of g. Then $\lim_{x\to a} \frac{f(x)}{g(x)}$ is
      i. 2
      ii. does not exist
      iii. not enough information
      iv. 3

Sample Honors Calculus II Worksheet

1. One of the most important functions in analysis is the gamma function,

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, x > 0.$$

   (a) Use integration by parts to prove that $\Gamma(x+1) = x\Gamma(x)$.

   (b) Show that $\Gamma(1) = 1$. Conclude that $\Gamma(n) = (n-1)!$ for all natural numbers n.
   The gamma function provides a simple example of a continuous function which interpolates the values of $n!$ for natural numbers x.

2. (a) Give an example of a sequence which is bounded above but diverges.

   (b) Give an example of a sequence bounded above by 1/2, below by -1/2, but which has no limit.

   (c) Give an example of a sequence $\{a_n\}$ such that $a_{n+1} > a_n > 0$ and $\lim_{n\to\infty} a_n = 2$.

   (d) Give an example of a sequence $\{a_n\}$ such that $a_{2n} > 1, a_{2n+1} < 1$ and $\lim_{n\to\infty} a_n = 1$.

3. Find two different ways to prove that $1 = .999....$ What does this imply about the uniqueness of numerical representation?

4. Is there a value of r for which the series

$$\Sigma_{k=0}^{\infty} r^k = \frac{7}{8}?$$

   Explain your answer.

5. Some copy machines will make reduced copies. Suppose you copy a page 8 inches wide and it comes out 3/4 as wide. Then you copy this, and so on, indefinitely. How far will the original and all the copies reach if you lay them out side by side on a long table?

6. You are driving along in Texas Hill Country and are experiencing car trouble when you come to the bottom of a steep hill. You begin to ascend the hill, but due to difficulties with your car, you begin rolling downhill. You manage to stop descending only after rolling downhill half the initial distance ascended. You start up again and ascend one third of the initial distance upward before your car acts up again and forces you downhill one fourth of the initial distance. This continues so that at the nth stage you are either rolling downhill one nth of the initial distance or moving uphill one nth of the initial distance. As this continues indefinitely, prove or disprove that you will have moved a finite distance uphill/downhill. Prove or disprove that the total distance traveled is finite. Do you ever make it up the hill?