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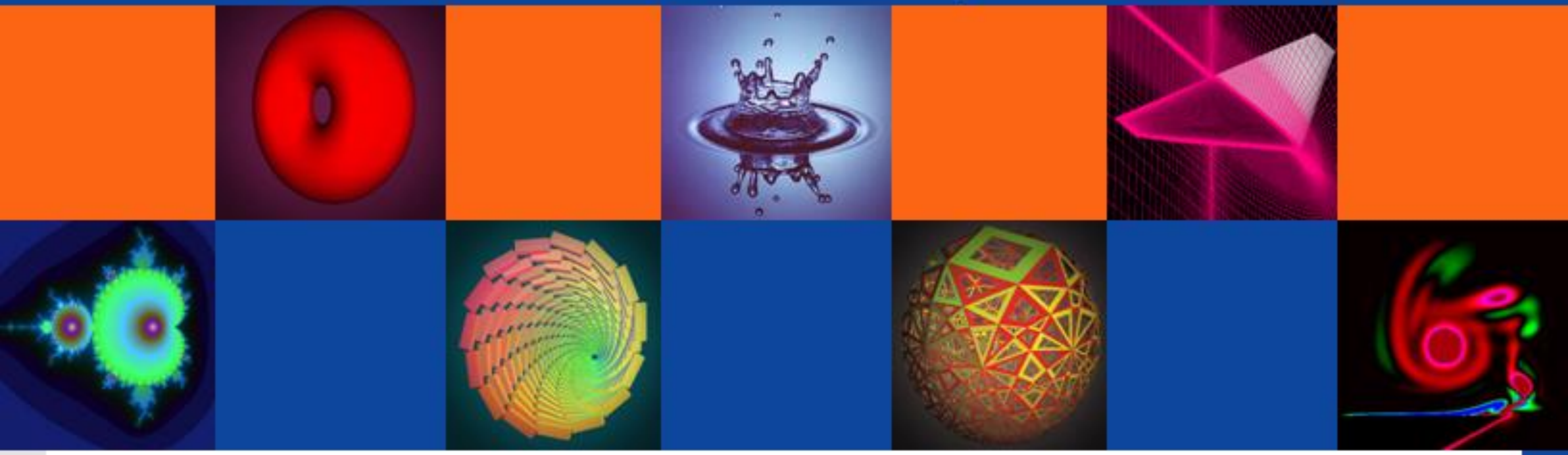

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Abstract

The Boussinesq eddy viscosity assumption (1872) **is still widely used in turbulence modeling although Reynolds stress transport model is considered. In this paper, we use DNS to study the Boussinesq assumption and found the Boussinesq eddy viscosity assumption is lack of scientific foundation. It may be viable in the second quarter (ejection as u'<0 and w'>0) and fourth quarter (sweep as u'>0 and w'<0) but fails even in sign which is not adjustable in the first quarter and third quarter. A counterexample is given for the case of high speed flow passing a micro vortex generator (MVG), where the flow separation appears. Both computation and experiment show there is no direct relation between Reynolds stress and the mean velocity gradients. We will try to use the DNS data to develop model for Reynolds stress.**

I. Introduction

The Reynolds averaged Navier-Stokes (RANS) equations are not self-closed and have many undetermined terms which are called Reynolds stress. These terms must be estimated by so-called turbulence models. Although the transport equations for Reynolds stress have been developed, they are not only more expensive, but also have to use additional models. Therefore, most turbulence models are still based on the Boussnesq eddy viscosity assumption, in which the Reynolds stress is directly co-related to the gradient or strain of the mean velocity. These eddy viscosity models with adjustable empirical coefficients like mixed-length, $k-, k-\varepsilon, k-\omega, etc.$ models are in general successful for flat plate and simple flow. The foundation of these models is based on the assumption that turbulence is a mixing process. This is partially correct since in turbulent flow, the rotation is dominant and so-called ejection and sweeps are widely observed. There must be some co-relation between flow fluctuation and gradient of mean flow gradient. However, as our DNS study showed, the real turbulent flow has much complicated vortex structure and is not easy to simply build up a simple relation between flow fluctuation (Reynolds stress in a time-average sense) and the mean flow gradient. In practice, these eddy viscosity models always fail for swirling flow and flow with separation. In this paper, we just provide some counterexamples to show the Boussinesq eddy viscosity assumption is lack of scientific foundation and further more accurate turbulence models are needed. These new turbulence models should be based on the vortex package structure and relative motion between these vortex packages. These new models can be obtained from DNS data base since u', v', w' p' are all obtained by DNS computation. These are the goal of our DNS study which is to develop more accurate turbulence model for general flow including the swirling flow and separated flow.

II Eddy Viscosity Models

2.1 Two Dimensional RANS Governing Equations

The time-averaging process, of obtaining mean quantities, is applied on the incompressible, two-dimensional equations of continuity and the *conservative* form of momentum and energy that produces the time-averaged governing equations or more popularly known as the Reynolds-Averaged Navier-Stokes (RANS) equations yields

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{1}$$

$$\begin{aligned}\frac{\partial \bar{u}}{\partial t} + \frac{\partial (\bar{u}\bar{u})}{\partial x} + \frac{\partial (\bar{v}\bar{u})}{\partial y} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial x}\left(\nu \frac{\partial \bar{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu \frac{\partial \bar{u}}{\partial y}\right) \\ &+ \frac{\partial}{\partial x}\left[\nu \frac{\partial \bar{u}}{\partial x}\right] + \frac{\partial}{\partial y}\left[\nu \frac{\partial \bar{v}}{\partial x}\right] - \left[\frac{\partial (\overline{u'u'})}{\partial x} + \frac{\partial (\overline{u'v'})}{\partial y}\right]\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial v}{\partial t} + \frac{\partial (\bar{u}\bar{v})}{\partial x} + \frac{\partial (\bar{v}\bar{v})}{\partial y} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} + \frac{\partial}{\partial x}\left(\nu \frac{\partial \bar{v}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu \frac{\partial \bar{v}}{\partial y}\right) \\ &+ \frac{\partial}{\partial x}\left[\nu \frac{\partial \bar{u}}{\partial y}\right] + \frac{\partial}{\partial y}\left[\nu \frac{\partial \bar{v}}{\partial y}\right] - \left[\frac{\partial (\overline{u'v'})}{\partial x} + \frac{\partial (\overline{v'v'})}{\partial y}\right]\end{aligned} \tag{3}$$

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial (\bar{u}\bar{T})}{\partial x} + \frac{\partial (\bar{v}\bar{T})}{\partial y} = \frac{\partial}{\partial x}\left(\frac{k}{\rho C_p}\frac{\partial \bar{T}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{k}{\rho C_p}\frac{\partial \bar{T}}{\partial y}\right) - \left[\frac{\partial \overline{u'T'}}{\partial x} + \frac{\partial \overline{v'T'}}{\partial y}\right] \tag{4}$$

where $\bar{u}$, $\bar{v}$, $\bar{p}$, $\bar{T}$ are mean values and u', v', p', T' are turbulent fluctuations. The term $k / \rho C_p$ in equation (4) is the thermal diffusivity α of the fluid. The equations above are similar to those formulated for laminar flows, except for the presence of additional terms of the form $\overline{a'b'}$. As a result, we have three additional unknowns (in three dimensions, we will have nine additional unknowns), known as the Reynolds stresses, in the time-averaged momentum equations. Similarly, the time-averaged temperature equation shows extra terms $\overline{u'T'}$ and $\overline{v'T'}$.

The time-averaged equations can be solved if the Reynolds stresses and extra temperature transport terms can be related to the mean flow and heat quantities.

2.2 Boussinesq Eddy Viscosity Assumption

It was proposed by Boussinesq (1872) that the Reynolds stresses could be linked to the mean rates of deformation.

$$-\rho\overline{u'u'} = 2\mu_T \frac{\partial \overline{u}}{\partial x} - \frac{2}{3}\rho k \qquad -\rho\overline{\upsilon'\upsilon'} = 2\mu_T \frac{\partial \overline{\upsilon}}{\partial y} - \frac{2}{3}\rho k$$

$$-\rho\overline{u'\upsilon'} = \mu_T \left(\frac{\partial \overline{\upsilon}}{\partial x} + \frac{\partial \overline{u}}{\partial y} \right) \tag{5}$$

The right hand side is analogous to *Newton's law of viscosity*, except for the appearance of the turbulent or eddy viscosity μ_T and turbulent kinetic energy k.

In Equation (5) the turbulent momentum transport is assumed to be proportional to the mean gradients of velocity. Similarly the turbulent transport of temperature is taken to be proportional to the gradient of the mean value of the transported quantity. In order words,

$$-\rho\overline{u'T'} = \Gamma_T \frac{\partial \overline{T}}{\partial x} \qquad -\rho\overline{v'T'} = \Gamma_T \frac{\partial \overline{T}}{\partial y} \tag{6}$$

where Γ_T is the turbulent diffusivity. Since the turbulent transport of momentum and heat is due to the same mechanisms – eddy mixing – the value of the turbulent diffusivity can be taken to be close to that of turbulent viscosity μ_T. Based on the definition of the turbulent Prandtl number Pr_T, we obtain

$$Pr_T = \frac{\mu_T}{\Gamma_T}$$

Experiments have established that this ratio is often nearly constant. Most CFD procedures assume this to be the case and use values of Pr_T around unity.

2.3 *k-ε model*

Since the complexity of turbulence in most engineering flow problems precludes the use of any simple formulae, it is possible to develop similar transport equations to accommodate the turbulent quantity k and other turbulent quantities one of which is the rate of dissipation of turbulent energy ε. Here we just introduce the form of a typical two-equation turbulence model that is commonly used in handling many turbulent fluid engineering problems, the *standard k-ε model* by Launder and Spalding (1974).

Some preliminary definitions are required first. The turbulent kinetic energy k and rate of dissipation of turbulent energy ε can be defined and expressed in Cartesian tensor notation as

$$k = \frac{1}{2} u'_i \, u'_i \qquad \text{and} \qquad \varepsilon = \nu_T \overline{\left(\frac{\partial u'_i}{\partial x_j}\right)\left(\frac{\partial u'_i}{\partial x_j}\right)} \qquad \text{where} \qquad i,j = 1,2,3$$

From the local values of k and ε, a local turbulent viscosity μ_T can be evaluated as

$$\mu_T = \frac{C_\mu \rho k^2}{\varepsilon} \tag{7}$$

and the kinematic turbulent or eddy viscosity is denoted by $\nu_T = \mu_T / \rho$.

By substituting the Reynolds stress expressions in equation (5) and the extra temperature transport terms in equation (6) into the governing equations (1), (2), (3) and (4), and removing the overbar that is by default indicating the average quantities, we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0 \tag{8}$$

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial (uu)}{\partial x} + \frac{\partial (\upsilon u)}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[(\nu + \nu_T)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\nu + \nu_T)\frac{\partial u}{\partial y}\right] \\ + \frac{\partial}{\partial x}\left[(\nu + \nu_T)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\nu + \nu_T)\frac{\partial \upsilon}{\partial x}\right] \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{\partial \upsilon}{\partial t} + \frac{\partial (u\upsilon)}{\partial x} + \frac{\partial (\upsilon\upsilon)}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[(\nu + \nu_T)\frac{\partial \upsilon}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\nu + \nu_T)\frac{\partial \upsilon}{\partial y}\right] \\ + \frac{\partial}{\partial x}\left[(\nu + \nu_T)\frac{\partial u}{\partial y}\right] + \frac{\partial}{\partial y}\left[(\nu + \nu_T)\frac{\partial \upsilon}{\partial y}\right] \end{aligned} \tag{10}$$

$$\frac{\partial T}{\partial t}+\frac{\partial(uT)}{\partial x}+\frac{\partial(\upsilon T)}{\partial y}=\frac{\partial}{\partial x}\left[\left(\frac{\nu}{Pr}+\frac{\nu_T}{Pr_T}\right)\frac{\partial T}{\partial x}\right]+\frac{\partial}{\partial y}\left[\left(\frac{\nu}{Pr}+\frac{\nu_T}{Pr_T}\right)\frac{\partial T}{\partial y}\right] \tag{11}$$

The term ν/Pr appearing in the temperature equation (11) is obtained from the definition of the laminar Prandtl number that is already defined in equation (1) as $Pr = \nu/\alpha$ where $\alpha = k/\rho\ C_p$. Interestingly, the time-averaged equations above have the same form as those developed for the laminar equations except for the additional turbulent viscosity found in the diffusion and non-pressure gradient terms for the momentum equations and also found in the diffusion term for the energy equation. Hence, the solution to turbulent flow in engineering problems entails greater diffusion that is imposed by the turbulent nature of the fluid flow.

The additional differential transport equations that is required for the standard k-ε model, for the case of a constant fluid property and expressed in non-conservation form, are

$$\frac{\partial k}{\partial t}+u\frac{\partial k}{\partial x}+\upsilon\frac{\partial k}{\partial y}=\frac{\partial}{\partial x}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial y}\right)+P-D \tag{12}$$

$$\frac{\partial \varepsilon}{\partial t}+u\frac{\partial \varepsilon}{\partial x}+\upsilon\frac{\partial \varepsilon}{\partial y}=\frac{\partial}{\partial x}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right)+\frac{\varepsilon}{k}\left(C_{\varepsilon 1}P-C_{\varepsilon 2}D\right) \tag{13}$$

where the production term P is formulated as

$$P=2\nu_T\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial \upsilon}{\partial y}\right)^2\right]+\nu_T\left(\frac{\partial u}{\partial y}+\frac{\partial \upsilon}{\partial x}\right)^2$$

and the destruction term D is given by ε. The physical significance of the above equations is: *the rate of change and the advection transport of k or ε equals the diffusion transport combined with the rate of production and destruction of k or ε.* The equations contain five adjustable constants C_μ, σ_k, σ_ε, $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$. These constants have been arrived at by comprehensive data fitting for a wide range of turbulent flows (Launder and Spalding, 1974):

$C_\mu = 0.09$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.3$, $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$.

The production and destruction of turbulent kinetic energy are always closely linked in the k-equation (12). The dissipation rate ε is large where the production of k is large. The model equation (12) assumes that the production and destruction terms are proportional to the production and destruction terms of the k-equation. Adoption of such terms ensures that ε increases rapidly if k increases rapidly and that it decreases sufficiently fast to avoid non-physical (negative) values of turbulent kinetic energy if k decreases. The factor ε / k in the production and destruction terms makes these terms dimensionally correct in the ε-equation (13). There are many other eddy viscosity models which are widely used in industry. In general, these models work well for flat plate and other simple flow but meet barrier when the flow separation appears and for swirling flow.

2.4 The Limitation of Eddy Viscosity Assumption

The eddy viscosity assumption given by Boussinesq

$$-\rho\overline{u'v'} = \mu_T\left(\frac{\partial\overline{v}}{\partial x}+\frac{\partial\overline{u}}{\partial y}\right)$$

is just assumption and does not have strict mathematical proof. On the other hand, by using adjustable empirical coefficients, the eddy viscosity may work well.

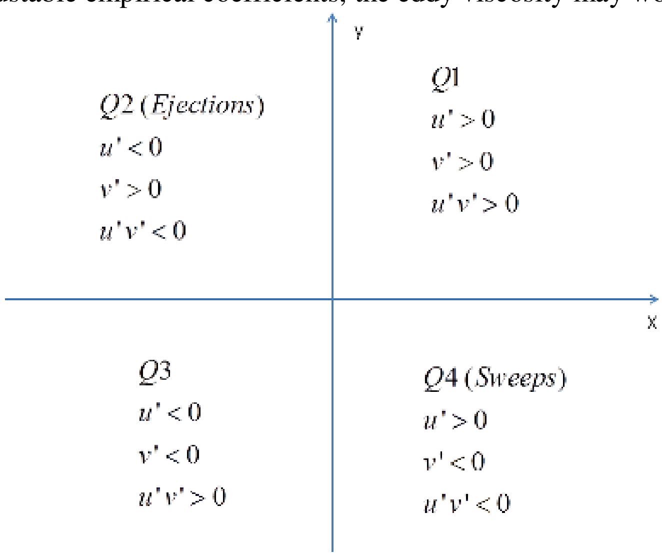


Figure 1: Flow fluctuations in different quarters

Apparently, the sign of Bousnessiq assumption coincides with the mean velocity gradient (Figure 1) in a boundary layer with Blasius type solution (Figure 2) which the perturbations are pretty much located in the second quarter (ejection) or the fourth quarter (sweeps). Any perturbations located in the first or third quarters directly violate the Bousnessiq assumption and become a counterexample.

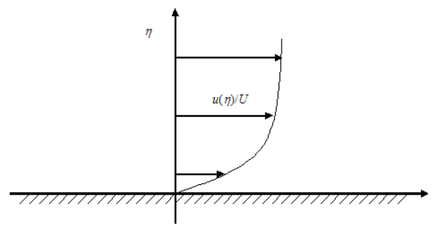
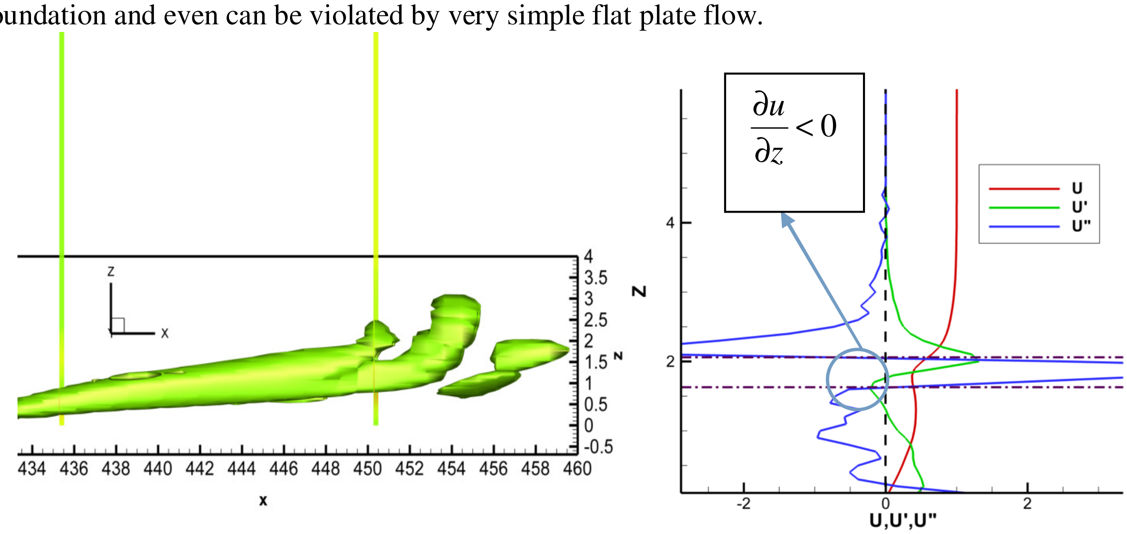


Figure 2: Blasius solution

However, in turbulent boundary layer, the instantaneous velocity or even the mean velocity could be very complicated. For a very simple transitional flow on a flat plate, a velocity and its derivatives profile is givenis Figure 3b, which is located at a cross vortex ring section. Although the perturbations are dominated with ejection (second quarter) and sweeps (Fourth quarter,) the first order derivative of the streamwise velocity, $\frac{\partial u}{\partial z} < 0$

in the middle position of the section. This directly opposes the Bousnessiq's assumption, $-\rho\overline{u'v'} = \mu_T\left(\frac{\partial \overline{v}}{\partial x} + \frac{\partial \overline{u}}{\partial y}\right)$ because the left hand side is positive but the right hand side is negative assuming μ_T must be positive. This clearly shows that the Bousnessiq assumption is lack of scientific foundation and even can be violated by very simple flat plate flow.



(a) Vortex rings (b) velocity profile and derivatives cross the ring

Figure 3: Velocity profile and derivatives with inflection points in a boundary layer

2.5 High Speed Counterexample to Bousnessiq Eddy Viscosity Assumption

A supersonic flow passing a micro vortex generator (MVG) is studied by high order LES and experiment with 3-D PIV technology (Figure 4.) The two approaches are compared very well (Sun et al, 2013.) The LES results are well validated by experiment. Figure 5(a) depicts the mean flow velocity profile after MVG and Figure 5(b) gave the corresponding Reynolds stress. Let us take a look at the section of *x/h=22.8*and check *y/h=2.5,*both $\overline{u'v'}$ and $\frac{\partial \overline{u}}{\partial y}$ are negative. Apparently, the Bousenessiq assumption: $-\rho\overline{u'v'} = \mu_T\left(\frac{\partial\bar{v}}{\partial x}+\frac{\partial\bar{u}}{\partial y}\right)$ cannot stand unless $\mu_T < 0$ which is impossible. From the two counterexamples described above, we can easily find that there is no direct co-relation between Reynolds stress and the gradient of mean flow velocity.

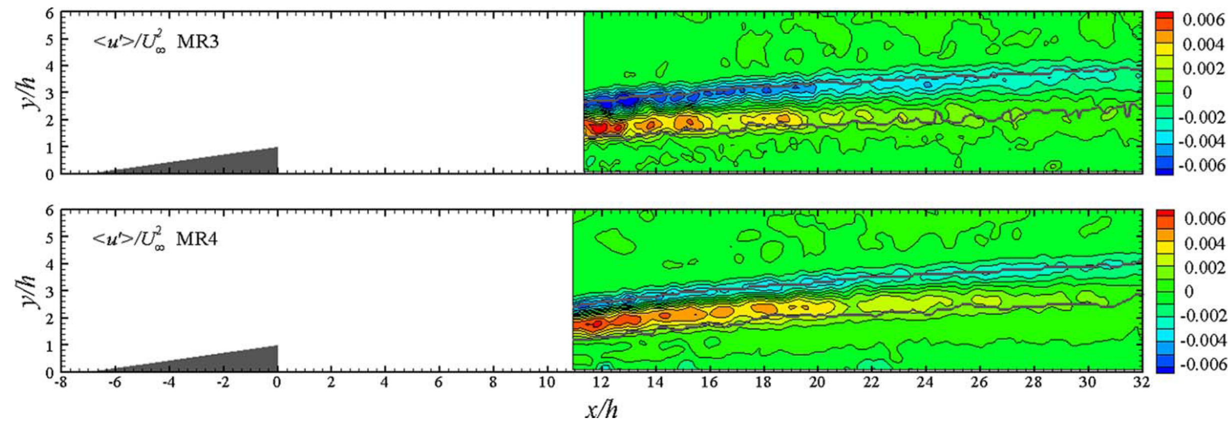
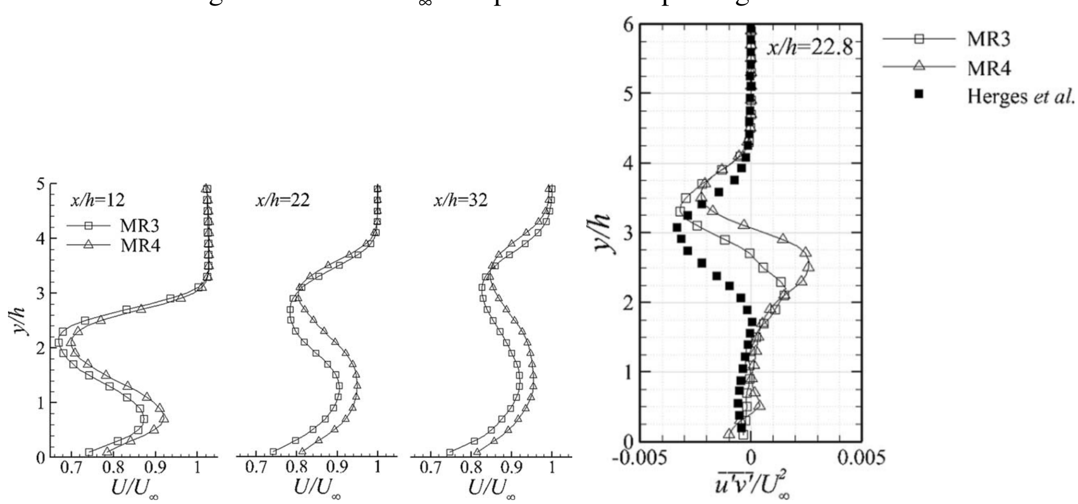


Figure 4: $<u'>/U_\infty^2$ in supersonic flow passing MVG



(a) **(b)**

FIG. 5: Comparison of profiles of Stremwise velocity U *and* $\overline{u'v'}/U_\infty^2$ *at x/h = 22.8*

III Important Conclusions

The Bousnessiq eddy viscosity assumption is lack of scientific foundation and even can be violated by a simple flat plate flow. This assumption will fail with swirling flow and flow with separation as shown by some counterexamples for high speed flow passing micro vortex generator. New turbulence model based on the DNS data set is needed.

IV Future work

We must develop new turbulence model based on DNS data base. More detailed description of the DNS results and analysis will be given in the final AIAA paper.

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